

MAU34103 - Lie algebras and Lie groups

Practice Sheet 1

Trinity College Dublin

Course homepage

These problems are just for practice, to help you warm up for the homework, and get more familiar with the material. I strongly encourage you to give them a try, as the best way to learn maths is through practice. They are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available alongside the problems

Exercise 1 *The special linear Lie algebra*

1. Consider the basis of $\mathfrak{sl}_2(\mathbb{R})$ given by

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Compute the brackets

$$[x, y], [h, x], \text{ and } [h, y]$$

and express your answer in terms of the basis.

2. Hence compute the brackets

$$[x, [x, y]], [h, [x, y]], [2x - 3h, [y + h, x]].$$

3. Determine the (3×3) matrices of the linear operators

$$\text{ad}_x, \text{ad}_y, \text{ad}_h : \mathfrak{sl}_2(\mathbb{R}) \rightarrow \mathfrak{sl}_2(\mathbb{R})$$

in this basis

Solution 1

1. We compute

$$\begin{aligned} [x, y] &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h \\ [h, x] &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2x \\ [h, y] &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} = -2y. \end{aligned}$$

2. We find

$$\begin{aligned} [x, [x, y]] &= [x, h] = -[h, x] = -2x \\ [h, [x, y]] &= [h, h] = 0 \\ [2x - 3h, [y + h, x]] &= 2[x, [y, x]] - 3[h, [y, x]] + 2[x, [h, x]] - 3[h, [h, x]] \\ &= -2[x, [x, y]] + 3[h, [x, y]] + 4[x, x] - 6[h, x] \\ &= 4x + 0 + 0 - 12x = -8x. \end{aligned}$$

3. We have that

$$\mathrm{ad}_x(x) = 0, \mathrm{ad}_x(y) = h, \mathrm{ad}_x(h) = -2x$$

and so

$$\mathrm{ad}_x \leftrightarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -2 & 0 & 0 \end{pmatrix}$$

Similarly

$$\mathrm{ad}_y \leftrightarrow \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

and

$$\mathrm{ad}_h \leftrightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Exercise 2 *Dimension computations*

Compute the dimension of the following Lie algebras as real vector spaces:

1. $\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$,
2. $\mathfrak{sl}_n(\mathbb{R}) = \{x \in M_n(\mathbb{R}) \mid \mathrm{tr}(x) = 0\}$,
3. $\mathfrak{so}_n(\mathbb{R}) = \{x \in M_n(\mathbb{R}) \mid x + x^T = 0\}$

Solution 2

Throughout, we will denote by $E_{i,j}$ the $n \times n$ matrix which has a 1 in position (i, j) and zeroes elsewhere. Note that these are linearly independent.

1. Clearly $\{E_{i,j}\}_{1 \leq i,j \leq n}$ forms a basis, and so $\dim \mathfrak{gl}_n(\mathbb{R}) = n^2$.
2. The set $\{E_{i,j}\}$ with $i \neq j$ form a linearly independent set in $\mathfrak{sl}_n(\mathbb{R})$, and span all trace-zero matrices with zeroes along the diagonal. Since every trace-zero matrix is a sum of such a matrix and a diagonal matrix, it is enough to find a basis for the space of trace-zero diagonal matrices. We claim $\{F_i = E_{i,i} - E_{i+1,i+1}\}_{1 \leq i \leq n-1}$ forms a basis of this subspace. These span the subspace: Suppose i is the smallest index such that

$M_{i,i} \neq 0$. Then subtracting $M_{i,i}F_i$ from M , we obtain a trace-zero matrix with fewer non-zero entries along the diagonal. We can repeat this until the only non-zero entries are the $(n-1, n-1)$ and (n, n) entries. These must sum to 0, so we have a multiple of F_{n-1} . Thus, the F_i span trace-zero diagonal matrices. Suppose

$$\sum \lambda_i F_i = 0.$$

Then, the coefficients of each $E_{i,i}$ must be 0. Hence $\lambda_1 = 0$, $\lambda_2 - \lambda_1 = 0$, and so on. We must then have $\lambda_i = 0$ for all i , and so the F_i are linearly independent.

Thus $\dim \mathfrak{sl}_n(\mathbb{R}) = (n^2 - n) + (n - 1) = n^2 - 1$.

3. The set $\{G_{i,j} = E_{i,j} - E_{j,i}\}_{i>j}$ are clearly elements of $\mathfrak{so}_n(\mathbb{R})$. It is straightforward to check that they form a basis, and hence

$$\dim \mathfrak{so}_n(\mathbb{R}) = \binom{n}{2}$$

as we have a basis element for every pair $\{i, j\}$ of distinct integers between 1 and n .

Exercise 3 *The Heisenberg Lie algebra*

Recall that we defined the Heisenberg Lie algebra by

$$\mathcal{H} = \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}q$$

equipped with Lie bracket

$$[f, q] = 0, [f, g] = \text{coeff}_{z^{-1}}(f'g)q$$

for all $f, g \in \mathbb{C}[z, z^{-1}]$.

1. Verify that this bracket is anticommutative, either showing

$$[f, f] = 0 \text{ or } [f, g] + [g, f] = 0.$$

2. Verify that this bracket satisfies the Jacobi identity

3. Show that

$$[z^m, z^n] = m\delta_{m,-n}q$$

where

$$\delta_{a,b} = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

4. Show that $\mathbb{C}[z]$ is a Lie subalgebra, and is, in fact, abelian.

Solution 3

1. Note that $f'f = \frac{1}{2}(f^2)'$. As there is no monomial z^n which differentiates to z^{-1} , the coefficient is therefore 0. This $[f, f] = 0$.

Similarly, as $f'g + fg' = (fg)'$, the coefficients of z^{-1} in $f'g$ and fg' must cancel. Hence

$$[f, g] + [g, f] = 0.$$

2. The Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

clearly holds if one of A, B, C is equal to q , as every term will be 0. So suppose each is a Laurent polynomial and consider

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]]$$

As $[f, g], [g, h], [h, f]$ are all scalar multiples of q , every term is again 0, and so the Jacobi identity holds.

- 3.

$$[z^m, z^n] = \text{coeff}_{z^{-1}}(mz^{m+n-1})q = \begin{cases} mq & \text{if } m+n=0 \\ 0 & \text{otherwise} \end{cases} = m\delta_{m,-n}q$$

4. Since $[z^m, z^n] = 0$ for all $m, n > 0$, and can be checked to be 0 if $m = 0$ or $n = 0$, $\mathbb{C}[z]$ forms an abelian Lie algebra and in particular a Lie subalgebra.

Exercise 4 *A more general Heisenberg algebra*

Given a vector space V , and a bilinear map $\omega : V \times V \rightarrow K$ such that

$$\omega(x, y) + \omega(y, x) = 0$$

we define the Heisenberg Lie algebra

$$\mathcal{H}_\omega := V \oplus Kq$$

with Lie bracket

$$[x, q] = 0, \quad [x, y] = \omega(x, y)q$$

for all $x, y \in V$.

1. Show $\mathcal{H}_\omega$ is a Lie algebra
2. Consider the Heisenberg Lie algebra for $V = \mathbb{C}x \oplus \mathbb{C}y$ with

$$\omega(x, y) = 1.$$

Show that this Lie algebra is isomorphic to the Lie algebra of 3×3 strictly upper triangular matrices:

$$\mathcal{H}_\omega \cong \mathfrak{n}_3.$$

3. Compute the derived Lie algebras $[\mathfrak{sl}_2(\mathbb{C}), \mathfrak{sl}_2(\mathbb{C})]$ and $[\mathfrak{n}_3, \mathfrak{n}_3]$. Hence conclude

$$\mathfrak{sl}_2(\mathbb{C}) \not\cong \mathcal{H}_\omega$$

Solution 4

1. Anticommutativity follows from anticommutativity of ω , so it suffices to show the Jacobi identity. The Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

clearly holds if one of A, B, C is equal to q , as every term will be 0. So suppose each is an element of v and consider

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]]$$

As $[f, g], [g, h], [h, f]$ are all scalar multiples of q , every term is again 0, and so the Jacobi identity holds. Thus $\mathcal{H}_\omega$ is a Lie algebra.

2. $\mathcal{H}_\omega$ has a basis x, y, q satisfying

$$[x, y] = q, [x, q] = [y, q] = 0.$$

The Lie algebra $\mathfrak{n}_3$ has a basis $E_{1,2}, E_{2,1}, E_{2,2}$ satisfying

$$[E_{1,2}, E_{2,1}] = E_{2,2}, [E_{1,2}, E_{2,2}] = [E_{2,1}, E_{2,2}] = 0.$$

Thus the map $x \mapsto E_{1,2}, y \mapsto E_{2,1}, q \mapsto E_{2,2}$ is a Lie algebra homomorphism. It is clearly bijective, giving an isomorphism.

3. The derived Lie algebra $[\mathfrak{sl}_2(\mathbb{C}), \mathfrak{sl}_2(\mathbb{C})]$ will be spanned by

$$[x, y] = h, \frac{1}{2}[h, x]x, \frac{-1}{2}[h, y] = y$$

and hence contains all of $\mathfrak{sl}_2(\mathbb{C})$.

The derived Lie algebra $[\mathfrak{n}_3, \mathfrak{n}_3]$ will be spanned by

$$[E_{1,2}, E_{2,1}] = E_{2,2}, [E_{1,2}, E_{2,2}] = [E_{2,1}, E_{2,2}] = 0$$

and is therefore equal to $\mathbb{C}E_{2,2}$.

If $f : \mathfrak{sl}_2(\mathbb{C}) \rightarrow \mathcal{H}$ we an isomorphism, we would have

$$f([\mathfrak{sl}_2(\mathbb{C}), \mathfrak{sl}_2(\mathbb{C})]) = [f(\mathfrak{sl}_2(\mathbb{C})), f(\mathfrak{sl}_2(\mathbb{C}))] = [\mathcal{H}, \mathcal{H}] \cong [\mathfrak{n}_3, \mathfrak{n}_3]$$

and hence

$$\mathfrak{sl}_2(\mathbb{C}) \cong \mathbb{C}E_{2,2}$$

which is impossible.

Exercise 5 *Very Witt-y*

Recall the Witt algebra

$$\text{Der } \mathbb{C}[t] = \mathbb{C}[t] \frac{d}{dt},$$

is a Lie algebra with bracket

$$\left[f \frac{d}{dt}, g \frac{d}{dt} \right] = (fg' - f'g) \frac{d}{dt}.$$

This has a basis

$$L_n = -t^{n+1} \frac{d}{dt}, \quad n \geq -1.$$

Show that

$$[L_m, L_n] = (m - n)L_{m+n}$$

Solution 5

This is a direct computation

$$[L_m, L_n] = (nt^{m+n+1} - mt^{m+n+1}) \frac{d}{dt} = (m - n)L_{m+n}.$$

Exercise 6 Algebraic exponentials

Let A be an algebra over $\mathbb{R}$, and suppose that $\delta : A \rightarrow A$ is a derivation such that there exists an $N > 0$ such that $\delta^N(x) = 0$ for all $x \in A$.

1. Show that

$$\delta^n(xy) = \sum_{i=0}^n \binom{n}{i} \delta^i(x) \delta^{n-i}(y)$$

for all $n \geq 0$.

2. Hence show that the map

$$\begin{aligned} \exp(\delta) : A &\rightarrow A \\ x &\mapsto \sum_{n \geq 0} \frac{\delta^n(x)}{n!} \end{aligned}$$

is an algebra homomorphism:

$$\exp(\delta)(xy) = \exp(\delta)(x) \exp(\delta)(y)$$

for all $x, y \in A$.

3. Conclude that $\exp(\delta)$ is an isomorphism.

Note: If A is a normed algebra, then we can replace the requirement that $\delta^N(x) = 0$ with $\|\delta(x)\| \leq C\|x\|$ for some constant C , so that the series converges.

Solution 6

1. We proceed by induction. The case $n = 1$ holds by definition. Suppose it holds for δ^{n-1} . Then

$$\begin{aligned}
\delta^n(xy) &= \delta(\delta^{n-1}(xy)) \\
&= \delta\left(\sum_{i=0}^{n-1} \binom{n-1}{i} \delta^i(x) \delta^{n-1-i}(y)\right) \\
&= \sum_{i=0}^{n-1} \binom{n-1}{i} \delta^{i+1}(x) \delta^{n-1-i}(y) + \delta^i(x) \delta^{n-i}(y) \\
&= \sum_{i=1}^n \binom{n-1}{i-1} \delta^i(x) \delta^{n-i}(y) + \sum_{i=0}^{n-1} \binom{n-1}{i} \delta^i(x) \delta^{n-i}(y) \\
&= \delta^n(x) \delta^0(y) + \sum_{i=1}^{n-1} \left(\binom{n-i}{i-1} + \binom{n-1}{i} \right) \delta^i(x) \delta^{n-i}(y) + \delta^0(x) \delta^n(y) \\
&= \sum_{i=0}^n \binom{n}{i} \delta^i(x) \delta^{n-i}(y)
\end{aligned}$$

as

$$\binom{n-1}{i-1} + \binom{n-1}{i} = \binom{n}{i}.$$

2. We have

$$\begin{aligned}
\exp(\delta)(xy) &= \sum_{n \geq 0} \frac{\delta^n(xy)}{n!} \\
&= \sum_{n \geq 0} \sum_{i=0}^n \frac{\delta^i(x) \delta^{n-i}(y)}{i!(n-i)!} \\
&= \sum_{i,j \geq 0} \frac{\delta^i(x)}{i!} \frac{\delta^j(y)}{j!} \\
&= \left(\sum_{i \geq 0} \frac{\delta^i(x)}{i!} \right) \left(\sum_{j \geq 0} \frac{\delta^j(y)}{j!} \right) \\
&= \exp(\delta)(x) \exp(\delta)(y)
\end{aligned}$$

where the rearranging of the sums is valid as all sums involved are finite.

3. It is enough to show that it is invertible. As δ and $-\delta$ will commute, we will have

$$\exp(\delta) \exp(-\delta) = \exp(\delta - \delta) = \exp(0) = id$$

as in lectures. Thus $\exp(-\delta)$ is an inverse, and so $\exp(\delta)$ is an isomorphism.

Exercise 7 *Two's an anomaly for orthogonal matrices*

Let V be an n -dimensional vector space over a field K , and let $\sigma : V \times V \rightarrow K$ be a non-degenerate bilinear form. By choosing an isomorphism

$$V \cong K^n$$

we can find an invertible $S \in M_n(K)$ such that

$$\sigma(v, w) = v^T S w \text{ for all } v, w \in V$$

1. Show that

$$\mathfrak{o}(V, \sigma) := \{x \in \text{End}(V) \mid \sigma(xv, w) + \sigma(v, xw) = 0 \text{ for all } v, w \in V\}$$

is isomorphic to

$$\mathfrak{o}(n, S) = \{x \in M_n(K) \mid x^T S + Sx = 0\}$$

2. Show that, if K has characteristic not equal to two, then

$$x \in \mathfrak{o}(n, S) \Rightarrow \text{tr}(x) = 0$$

3. Now suppose that K has characteristic 2, and consider the case of $S = I$. Determine a basis of

$$\mathfrak{o}(2) = \{x \in M_2(K) \mid x + x^T = 0\}$$

and of

$$\mathfrak{so}(2) = \{x \in \mathfrak{o}(2) \mid \text{tr}(x) = 0\}.$$

Conclude that

$$\mathfrak{so}(2) \subset \mathfrak{o}(2)$$

is a strict inclusion.

Solution 7

1. If $\sigma(v, w) = v^T S w$, then

$$0 = \sigma(xv, w) + \sigma(v, xw) = v^T (x^T S + Sx) w$$

for all $v, w \in V$. In particular, choosing v and w to run over the basis, we must have that each entry of $x^T S + Sx$ is 0, and so

$$x^T S + Sx = 0.$$

Conversely, if $x^T S + Sx = 0$, then

$$\sigma(xv, w) + \sigma(v, xw) = 0$$

for all $v, w \in V$. Thus, under the identification of $\text{End}(V)$ and $M_n(K)$ via the isomorphism $V \cong K^n$, we get the desired isomorphism.

2. As S is invertible, $x^T S + Sx = 0$ is equivalent to

$$x^T + SxS^{-1} = 0 \quad \Rightarrow \quad \text{tr}(x^T) + \text{tr}(SxS^{-1}) = 0 \quad \Rightarrow \quad 2 \text{tr}(x) = 0.$$

Hence, the claim follows if K has characteristic not equal to 2.

3. We have that $x \in \mathfrak{o}(2)$ if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & c \\ b & d \end{pmatrix} = 0$$

This imposes that $b + c = 0$, but nothing else. It is then easy to check that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

forms a basis.

If we impose the trace-zero condition, then we also need $a + d = 0$, and so we find a basis given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

from which it is clear to see that

$$\mathfrak{so}(2) \subset \mathfrak{o}(2)$$

is a strict inclusion for dimensional reasons.