MAU34103 - Lie algebras and Lie groups

Practice Sheet 4

Trinity College Dublin

Course homepage

These problems are just for practice, to help you warm up for the homework, and get more familiar with the material. I strongly encourage you to give them a try, as the best way to learn maths is through practice. They are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available alongside the problems

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1 Representations of $\mathfrak{sl}_2$

In all the follows, we will be working over $\mathbb{C}$ unless otherwise specified.

Exercise 1

Let $V(n)$ be the unique $n+1$-dimensional irreducible representation of $\mathfrak{sl}_2$.

i) Determine the weights (h-eigenvalues) with multiplicity of $V(2) \otimes V(3)$,

ii) Hence determine decomposition of $V(2) \otimes V(3)$ into irreducible components.

iii) Determine a basis of $V(2) \otimes V(3)$ compatible with this decomposition.

Exercise 2

i) Determine the weights (h-eigenvalues) with multiplicity of $\mathrm{Hom}(V(2), V(3))$,

ii) Hence determine decomposition of $\mathrm{Hom}(V(2), V(3))$ into irreducible components.

iii) Determine a basis of $\mathrm{Hom}(V(2), V(3)$ compatible with this decomposition.

Exercise 3

For all $n \geq 0$, show that the dual representation $V(n)^*$ is isomorphic (as a representation) to $V(n)$.

Hint: A finite dimensional representation is entirely determined by the multiplicities of its weights

Exercise 4

Let V, W be finite dimensional $\mathfrak{sl}_2$-representations and suppose

$$V = \bigoplus_{i \in \mathbb{Z}} V_i, \ W = \bigoplus_{j \in \mathbb{Z}} W_j$$

are their weight space decompositions (h-eigenspace decomposition). One can show that, as vector spaces,

$$V \otimes W \cong \bigoplus_{i,j} V_i \otimes W_j, \ \mathrm{Hom}(V, W) \cong \bigoplus_{i,j} \mathrm{Hom}(V_i, W_j).$$

i) Show that $V_i \otimes W_j$ is a h-eigenspace with weight $i + j$,

ii) Show that $\mathrm{Hom}(V_i, W_j)$ is a h-eigenspace with weight $j - i$.

iii) Hence determine the decomposition of $V(m) \otimes V(n)$ into irreducibles,

iv) Similarly determine the decomposition of $\mathrm{Hom}(V(m), V(n))$ into irreducibles.

Exercise 5

Let V be a finite dimensional $\mathfrak{sl}_2$-representation and suppose

$$V = \bigoplus_{i \in \mathbb{Z}} V_i$$

is the weight space decomposition. Then, one can show that, as vector spaces,

$$V^* \cong \bigoplus_{i \in \mathbb{Z}} V_i^*.$$

i) Show that V_i^* is a h-eigenspace with weight $-i$,

ii) Hence show that $V \cong V^*$ as representations.

2 Unimodal sequences and an exotic representation

A (possibly infinite) sequence $\{a_n\}_{m_1 \leq n \leq m_2}$ is called *unimodal* if there exists $m_1 \leq m \leq m_2$ such that

$$a_{m_1} \leq a_{m_1+1} \leq \cdots \leq a_m \geq \cdots \geq a_{m_2-1} \geq a_{m_2}$$

or

$$a_{m_1} \geq a_{m_1+1} \geq \cdots \geq a_m \leq \cdots \leq a_{m_2-1} \leq a_{m_2}.$$

That is, it is a sequence that increases (or decreases) to an extreme value and then decreases (or increases) beyond it.

Exercise 6

Let V be a finite dimensional $\mathfrak{sl}_2$-representation, and let $V = \oplus_{i \in \mathbb{Z}} V_i$ be its weight decomposition. Show that, up to omitting 0, the sequences

$$a_n = \dim V_{2n}, \ b_n = \dim V_{2n+1}$$

are unimodal with extreme values a_0 and b_0.

Exercise 7

A *labelled simple graph* on n vertices is a collection

$$G \subset \{\{i,j\} \mid 1 \leq i \neq j \leq n\}$$

of 2-element subsets of $\{1,\ldots,n\}$, called *edges*.

Let W_n denote the $\mathbb{C}$-vector space with basis $\{e_G\}$ as G runs over all labelled simple graphs on n vertices.

For every $1 \leq i < j \leq n$, define linear operators

$$a_{ij}(e_G) := \begin{cases} e_{G\cup\{i,j\}} \text{ if } \{i,j\} \text{ is not an edge of } G, \\ 0 \text{ otherwise,} \end{cases}$$

$$b_{ij}(e_G) := \begin{cases} e_{G\setminus\{i,j\}} \text{ if } \{i,j\} \text{ is an edge of } G, \\ 0 \text{ otherwise.} \end{cases}$$

i) Show that if $\{i,j\} \neq \{k,\ell\}$, then

$$[a_{ij}, a_{k\ell}] = [a_{ij}, b_{k\ell}] = [b_{ij}, b_{k\ell}] = 0.$$

ii) Show that

$$[a_{ij}, b_{ij}](e_G) \begin{cases} e_G \text{ if } \{i,j\} \in G, \\ -e_G \text{ otherwise.} \end{cases}$$

iii) Define

$$X := \sum_{1\leq i<j\leq n} a_{ij}, \; Y := \sum_{1\leq i<j\leq n} b_{ij}$$

and

$$H(e_G) := \left(2k - \binom{n}{2}\right) e_G$$

where $k = |G|$ is the number of edges. Show that the map

$$\begin{aligned} \mathfrak{sl}_2 &\to \mathfrak{gl}(W_n) \\ x &\mapsto X, \\ y &\mapsto Y, \\ h &\mapsto H \end{aligned}$$

equips W_n with the structure of an sln_2-representation.

iv) Determine the dimensions of the weight spaces and hence conclude that the sequence $\{\binom{\binom{n}{2}}{k}\}_{0\leq k\leq \binom{n}{2}}$ is unimodal.

Exercise 8

The set of graphs comes with a natural action of the symmetric group S_n, where

$$\sigma(G) = \{\{\sigma(i), \sigma(j)\} \mid \{i, j\} \in G\}.$$

An equivalence class of labelled graphs under this action is called a *unlabelled simple graph* Let $g_{n,j}$ denote the number of unlabelled simple graphs with n vertices and k edges.

i) Show that the action of S_n on W_n given by $\sigma(e_G) = e_{\sigma(G)}$ commutes with the action of $\mathfrak{sl}_2$ from the prior exercise.

ii) Hence conclude that W_n/S_n is an $\mathfrak{sl}_2$-representation and determine its weight spaces.

iii) Conclude that $\{g_{n,k}\}_{k\in\mathbb{Z}}$ is unimodal.

3 Some infinite dimensional representations

The theory of infinite dimensional $\mathfrak{sl}_2$-representations is similarly nice, though we have to be much more careful about the possible weights, as we can obtain arbitrary complex numbers as h-eigenvalues. As such, we often want to restrict to "nice" representations, even in the infinite dimensional case.

This section is beyond the scope of the course and is purely for interest

Exercise 9

Let $\mathcal{H}$ denote the space of holomorphic functions in one variable.

i) Show that

$$x \cdot f = \frac{e^{2\pi i z}}{2\pi i} f'(z),\ y \cdot f = -\frac{e^{-2\pi z}}{2\pi i} f'(z),\ h \cdot f = \frac{1}{\pi i} f'(z)$$

equips $\mathcal{H}$ with the structure of an $\mathfrak{sl}_2$-module.

ii) Determine the set of h-eigenvalues and their corresponding eigenvectors.

iii) Show that x and y map between eigenspaces as follows:

$$x : \mathcal{H}_\lambda \to \mathcal{H}_{\lambda+2},$$
$$y : \mathcal{H}_\lambda \to \mathcal{H}_{\lambda-2}$$

Exercise 10

Let $\mathcal{P}$ denote the space of 1-periodic holomorphic functions in one variable.

i) Show that

$$x \cdot f = \frac{e^{2\pi i z}}{2\pi i} f'(z),\ y \cdot f = -\frac{e^{-2\pi z}}{2\pi i} f'(z),\ h \cdot f = \frac{1}{\pi i} f'(z)$$

equips $\mathcal{P}$ with the structure of an $\mathfrak{sl}_2$-module.

ii) Determine the set of h-eigenvalues and their corresponding eigenvectors.

iii) Show that x and y map between eigenspaces as follows:

$$x : \mathcal{H}_\lambda \to \mathcal{H}_{\lambda+2},$$
$$y : \mathcal{H}_\lambda \to \mathcal{H}_{\lambda-2}$$

iv) If you know any functional analysis or harmonic analysis, is there any sense in which the eigenvectors form a basis?

Exercise 11

The space $\mathcal{QM}$ of quasimodular forms is an algebra generated by

$$E_{2n}(q) = c_{2n} + \sum_{n \geq 1} \sigma_{2k-1}(n) q^n$$

where

$$\sigma_k(n) := \sum_{d|n} d^k$$

is a divisor power sum function, and c_{2n} is a constant. Modular and quasi-modular forms play a substantial role in number theory and the study of Diophantine equations, and so their structure if of central interest.

One can show

$$\mathcal{QM} \cong \mathbb{C}[E_2, E_4, E_6]$$

allowing us to view quasimodular forms as a polynomial algebra in 3 variables. We can equip this with an $\mathfrak{sl}_2$ structure by setting

$$\begin{aligned} x(f(E_2, E_4, E_6) &:= q\frac{d}{dq}(f(E_2, E_4, E_6) \\ y(f(E_2, E_4, E_6) &= -12\frac{\partial}{\partial E_2} f(E_2, E_4, E_6) \\ y(f(E_2, E_4, E_6) &= \left(2\deg_{E_2}(f) + 4\deg_{E_4}(f) + 6\deg_{E_6}(f)\right) f(E_2, E_4, E_6). \end{aligned}$$

The x-action is given more explicitly by setting

$$x(E_2) = \frac{E_2^2 - E_4}{12},\ x(E_4) = \frac{E_2E_4 - E_6}{3},\ x(E_6) = \frac{E_2E_6 - E_4^2}{2}$$

and having x act as a derivation on $\mathbb{C}[E_2, E_4, E_6]$.

Show that every quasimodular form $F = f(E_2, E_4, E_6)$ (other than $F = E_2^r$) such that $h \cdot F = kF$ and $\deg_{E_2}(f) = r$ can be written in the form

$$F = \sum_{j=0}^{r} x^j F_j$$

where $F_j = f_j(E_4, E_6)$ has $hF_j = k - 2j$.

Hint: Induct on the E_2-degree. Is there a constant C such that $F - CxyF$ has lower degree?