MAU34103 - Lie algebras and Lie groups

Practice Sheet 3

Trinity College Dublin

Course homepage

These problems are just for practice, to help you warm up for the homework, and get more familiar with the material. I strongly encourage you to give them a try, as the best way to learn maths is through practice. They are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available alongside the problems

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1 Representations of Lie algebras

Exercise 1

Recall that a representation of a Lie algebra L is defined as a Lie algebra homomorphism

$$\rho : L \to \mathfrak{gl}(V)$$

to the endomorphisms of a vector space V. The structure of an L-module on V is given by a bilinear map $L \times V \to V$ such that

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$$

for all $x, y \in L$, $v \in V$.

Convince yourself that an L-representation (V, ρ) and an L-module V are equivalent structures.

Exercise 2

Define the adjoint representation of a Lie algebra L by the map

$$\rho : L \to \mathfrak{gl}(L)$$

where $\rho(x)(y) := [x, y]$ for all $x, y \in L$.

i) Verify that this is indeed a Lie algebra representation, i.e. that ρ is a Lie algebra homomorphism.

ii) In the case of $L = \mathfrak{sl}_2(\mathbb{C})$, determine the matrices of $\rho(x), \rho(y), \rho(h)$ in the basis $\{x, y, h\}$ of L.

iii) Verify explicitly that these matrices satisfy

$$[\rho(u), \rho(v)] = \rho([u, v])$$

for all $u, v \in \{x, y, h\}$.

Exercise 3

Let $L = \mathfrak{sl}_2(\mathbb{C})$ be the special linear Lie algebra and define a representation

$$\rho : L \to \mathfrak{gl}(\mathbb{C}^3)$$

by

$$\begin{aligned} \rho(x)(e_1) = 0,\ \rho(x)(e_2) = e_3,\ \rho(x)(e_3) = -2e_1, \\ \rho(y)(e_1) = -e_3,\ \rho(y)(e_2) = 0,\ \rho(y)(e_3) = 2e_2, \\ \rho(h)(e_1) = 2e_1,\ \rho(h)(e_2) = -2e_2,\ \rho(h)(e_3) = 0. \end{aligned}$$

i) Determine the matrices of $\rho(x), \rho(y), \rho(h)$ in the standard basis

ii) Show that $(\mathbb{C}^3, \rho)$ is isomorphic to the adjoint representation (L, ad).

iii) Classify by hand all homomorphisms of representations

$$f : (\mathbb{C}^3, \rho) \to (L, \mathrm{ad})$$

Hint: We must have that $f(\rho(h)v) = \mathrm{ad}_h(f(v))$. *Consider eigenvectors!*

Exercise 4

Let (V, ρ) be an L-representation over a field K.

i) Let $V^* = \mathrm{Hom}(V, K)$. Define an action of L on V^* by

$$(x \cdot f)(v) := -f(x \cdot v)$$

Show that this defines the structure of an L-representation on V^*.

ii) Denote this dual representation by (V^*, ρ^*). Suppose that $\rho(x) \in \mathfrak{gl}(V)$ is given by a matrix $A(x)$ in some basis of V. Show that $\rho^*(x)$ has matrix $-A(x)^T$ in the dual basis of V^*.

Exercise 5

The vector space $V = \mathbb{C}^2$ can be viewed as a representation of any Lie subalgebra of $\mathfrak{gl}_2(\mathbb{C})$ via matrix multiplication.

i) View V as a representation of $\mathfrak{gl}_2(\mathbb{C})$. Show V^* is not isomorphic to V as a representation of $\mathfrak{gl}_2(\mathbb{C})$.

ii) View V as a representation of $\mathfrak{sl}_2(\mathbb{C})$. Show V^* *is* isomorphic to V as a representation of $\mathfrak{sl}_2(\mathbb{C})$.

Exercise 6

Let V, W be two L-representations over a field K

i) Show that the action of L on $\mathrm{Hom}(V, W)$ given by

$$(x \cdot f)(v) = x \cdot (f(v)) - f(x \cdot v)$$

defines the structure of an L-representation/module on $\mathrm{Hom}(V, W)$

ii) Show that the map

$$\begin{aligned} V^* \otimes W &\to \mathrm{Hom}(V, W), \\ f \otimes w &\mapsto (v \mapsto f(v)w) \end{aligned}$$

is an isomorphism of L-representations.

2 Solvable and nilpotent Lie algebras

Exercise 7

Let $\mathfrak{b}_3 \subset \mathfrak{gl}_3(\mathbb{C})$ be the Lie algebra of upper triangular matrices.

i) Find the central and derived series of $\mathfrak{b}_3$

ii) Determine if $\mathfrak{b}_3$ is nilpotent, solvable, or neither.

Exercise 8

Recall that any two-dimensional non-abelian Lie algebra L is isomorphic to the Lie algebra with basis $\{x, y\}$ and bracket determined by $[x, y] = x$. Is this Lie algebra nilpotent, solvable, or neither?

Exercise 9

i) If $f : L \to M$ is a surjective Lie algebra homomorphism, show that $f(\operatorname{rad} L) \subset \operatorname{rad}(M)$,

ii) If L, M are Lie algebras, show that $\operatorname{rad}(L \oplus M) = \operatorname{rad}(L) \oplus \operatorname{rad}(M)$,

iii) If $I \subset L$ is a solvable ideal, show that $\operatorname{rad}(L/I) \cong \operatorname{rad}(L)/I$.

Exercise 10

Show that $\mathfrak{gl}_n(\mathbb{C})$ is not solvable for any $n \geq 0$ by computing the derived algebra of $\mathfrak{gl}_n(\mathbb{C})$ explicitly. Hence conclude $\mathfrak{sl}_n(\mathbb{C})$ is not solvable.

Exercise 11

Determine a non-zero Lie algebra $L \subset \mathfrak{gl}_n(\mathbb{C})$ such that L is nilpotent, but L contains a matrix x such that $x^N \neq 0$ for any $N > 0$.

3 Simple and semisimple Lie algebras

Exercise 12

Show directly that $\mathfrak{sl}_2$ is a simple Lie algebra over a field of characteristic 0.

Exercise 13

Define a bilinear form on $\mathfrak{gl}_n(\mathbb{C})$ by

$$(x, y) := \operatorname{tr}(xy)$$

i) Show that $(-,-)$ is non-degenerate,

ii) Show that $\mathfrak{sl}_n(\mathbb{C})$ is orthogonal to $\mathbb{C}\,\mathrm{id}$ with respect to this form,

iii) Show that $(-,-)|_{\mathfrak{sl}_n}$ is non-degenerate and conclude $\mathfrak{sl}_n(\mathbb{C})$ is reductive,

Exercise 14

Using that L is reductive if it admits a faithful representation (V, ρ) such that the bilinear form

$$(x, y) := \operatorname{tr}_V(\rho(x)\rho(y)$$

is non-degenerate, show that $\mathfrak{so}_n(K)$, $\mathfrak{sp}_{2n}$, and $\mathfrak{su}(n)$ are semisimple.

Exercise 15

i) Using freely the properties of the Killing form given below, show that

$$\mathfrak{so}_4(\mathbb{C}) = \{x \in \mathfrak{gl}_4(\mathbb{C}) \mid x + x^T = 0\}$$

is semisimple.

Hint: The Lie algebra has basis $F_{12}, F_{13}, F_{14}, F_{23}, F_{24}, F_{34}$ where F_{ij} has (k, ℓ) entry 1 if $i = k, j = \ell$; -1 if $i = \ell, j = k$; and 0 otherwise. The Killing form satisfies

$$\kappa(X, Y) = 2\operatorname{tr}(XY)$$

where XY is the product of X and Y as matrices.

ii) An ideal of the Lie algebra $\mathfrak{so}_4(\mathbb{C})$ is

$$I = \{\alpha A + \beta B + \gamma C \mid \alpha, \beta, \gamma \in \mathbb{C}\}$$

where

$$\begin{aligned} A &= i(F_{12} + F_{34}), \\ B &= \frac{i}{2}\left(F_{13} - F_{24} + iF_{14} + iF_{23}\right), \\ C &= \frac{i}{2}\left(F_{13} - F_{24} - iF_{14} - iF_{23}\right). \end{aligned}$$

Determine $I^{\perp}$ with respect to the Killing form.

Fun Fact! The only 3*-dimensional simple Lie algebra over* $\mathbb{C}$ *is* $\mathfrak{sl}_2(\mathbb{C})$*, so we can conclude* $I \cong I^{\perp} \cong \mathfrak{sl}_2(\mathbb{C})$ *and* $\mathfrak{so}_4(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$