

MAU34103 - Lie algebras and Lie groups

Practice Sheet 2

Trinity College Dublin

Course homepage

These problems are just for practice, to help you warm up for the homework, and get more familiar with the material. I strongly encourage you to give them a try, as the best way to learn maths is through practice. They are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available alongside the problems

1 Logarithms and exponentials

Exercise 1 *Exponential equations*

In lectures, we saw that, for $x, y \in \mathfrak{gl}_n(\mathbb{R})$, $\exp(x)\exp(y) = \exp(x+y)$ if $[x, y] = 0$. Let's explore this further and see if we can start to determine a general formula for the product.

- i) Let $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Compute $e^x e^y$ and e^{x+y} .
- ii) Let $x, y \in \mathfrak{gl}_n(\mathbb{R})$ now be arbitrary. Suppose that

$$[x, [x, y]] = [y, [x, y]] = 0$$

Compute $e^{tx} y e^{-tx}$.

Hint: This is $\text{Ad}_{e^{tx}}(y)$. Convert this to a computation with the differential

iii) Hence show that

$$f(t) = e^{tx} e^{ty}, \quad \text{and} \quad F(t) = e^{tx+ty+\frac{t^2}{2}[x,y]}$$

satisfy the same differential equation. Conclude that

$$e^x e^y = e^{x+y+\frac{1}{2}[x,y]}$$

Remark. *This is a special case of the Baker-Campbell-Hausdorff formula, knowing in quantum mechanics as the disentangling theorem. The general formula is that $e^x e^y = e^z$ with*

$$z = x + y + \frac{1}{2}[x, y] + \frac{1}{12}[x, [x, y]] + \frac{1}{12}[y, [y, x]] + \cdots$$

It is not too hard to verify this up to any given length of brackets, by explicitly expanding out $e^{tx} e^{ty} = e^{tz}$, but proving it in general requires either some clever combinatorics or clever uses of ODEs

Exercise 2 A logarithmic failure

Recall that, for a linear group G , $x \in L(G)$ if $e^{tx} \in G$ for all $t \in \mathbb{R}$. We will see that it is not sufficient to have just $e^x \in G$.

i) Determine non-zero $x \in \mathfrak{gl}_2(\mathbb{R})$ such that $e^x = I$

Hint: In the complex numbers $e^{2\pi i} = 1$. Do we have a map $f : \mathbb{C} \rightarrow \mathfrak{gl}_2(\mathbb{R})$ such that $f(e^z) = e^{f(z)}$?

ii) Given an example of a linear group $G \subset \text{GL}_2(\mathbb{R})$ such that $I \in G$, but $x \notin L(G)$ for this x .

Hint: Think lower dimensional

Exercise 3 1-parameter subgroup (or general exponentiation)

Given a Lie group G , we call a Lie group homomorphism $\gamma : \mathbb{R} \rightarrow G$ (and its image) a *1-parameter subgroup*. Let $\mathfrak{g} = T_I G$ be the Lie algebra given by the tangent space to the identity. Note that, by choosing a chart around I , we may assume $\mathfrak{g} \subset \mathfrak{gl}_n(\mathbb{R})$ for some n . Given a chart (U, ϕ) you may also assume that $\phi(gh) = \phi(g)\phi(h)$ for all $g, h \in U$ such that $gh \in U$.

- i) If G is a linear group, and $x \in L(G)$, show that $t \rightarrow e^{tx}$ is a 1-parameter subgroup of G .
- ii) Show that a 1-parameter subgroup is determined by $\gamma(t)$ for $t \in [0, n^{-1})$ for any $n \geq 1$.
- iii) Show that any 1-parameter subgroup satisfies a differential equation near $t = 0$, given in a chart (U, ϕ) around I by

$$(\phi \circ \gamma)'(t) = x(\phi \circ \gamma)(t)$$

for some $x \in \mathfrak{g}$.

Hint: Differentiate the definition of a homomorphism

- iv) Show that, for any $x \in L(G)$, a solution to

$$(\phi \circ \gamma)'(t) = x(\phi \circ \gamma)(t)$$

in a chart around I satisfies

$$\gamma(s+t) = \gamma(s)\gamma(t)$$

for $s, t \in (-\varepsilon, \varepsilon)$ for some sufficiently small $\varepsilon > 0$.

Hint: A differential equation of this form has a unique solution given fixed initial conditions

- v) Hence conclude that, for any tangent vector x , there is a unique 1-parameter subgroup γ such that $\gamma'(0) = x$ (in some chart).

Such subgroups are how we define $\exp : \mathfrak{g} \rightarrow G$ in general. Namely, we define $\exp(x) := \gamma_x(1)$, where $\gamma_x(t)$ is the unique 1-parameter subgroup determined by x .

- vi) Show that $\exp((s+t)x) = \exp(sx)\exp(tx)$.

Hint: Chain rule and uniqueness of solutions to ODEs should let you compute $\exp(tx)$ explicitly in terms of γ_x .

Remark. For those of you more familiar with differential geometry, a tangent vector $x \in T_I G$ determines a vector field X such that $X_g(f) = X(f \circ g)$ for all $g \in G$ and $f : G \rightarrow \mathbb{R}$. Such a vector field is called left-invariant, as they are invariant under the push forward associated to the left multiplication

map $L_g : G \rightarrow G$. The Peano Existence Theorem implies there is a unique integral curve γ through I associated to X . As, for any given s , $\gamma(s+t)$ and $\gamma(s)\gamma(t)$ are both integral curves through the same point, they must be equal, and so the integral curve is a 1-parameter subgroup.

2 Induced algebra homomorphisms

Exercise 4 A complex correspondence

Consider the Lie groups

$$U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$$

$$SO(2) = \{A \in GL_2(\mathbb{R}) \mid A^T A = I\}$$

- i) Show that the Lie algebra (over $\mathbb{R}$) of $U(1)$ is

$$\mathfrak{u}(1) = \{z \in \mathbb{C} \mid z + \bar{z} = 0\}.$$

- ii) Show that the map

$$U(1) \rightarrow SO(2)$$

$$e^{it} \mapsto \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

is an isomorphism.

- iii) Determine the induced isomorphism

$$\mathfrak{u}(1) \rightarrow \mathfrak{so}_2(\mathbb{R}).$$

Exercise 5 An induced isomorphism

Consider the homomorphism of Lie groups

$$f : (\mathbb{R}, +) \rightarrow (U(1), \cdot)$$

$$t \mapsto e^{it}.$$

- i) Explain, without any computation, why the induced Lie algebra homomorphism must be an isomorphism.
- ii) Determine explicitly the induced Lie algebra homomorphism.
- iii) Determine $\ker(f)$ and based on its topological properties, make a wild conjecture about when two Lie groups will have the same Lie algebra.

3 Subgroups and quotient groups

The following questions increasingly lean into the geometry of Lie groups, which we have not spent much time on, and so should be considered on the more challenging side

Exercise 6 *A local almost group*

Let G be a Lie group with identity id . We will show that, for any open $U \ni \text{id}$, there exists an open $V \subset U$ containing id such that

$$V = V^{-1} = \{g^{-1} \mid g \in V\}, \quad V^2 = \{gh \mid g, h \in V\} \subset U.$$

- i) Show that, for any open $U \subset G$, the set

$$U^{-1} = \{g^{-1} \mid g \in U\}$$

is open. Show that, if $U \ni \text{id}$, then there exists open $V \subset U$ containing id such that $V = V^{-1}$.

Hint: The map $g \rightarrow g^{-1}$ is smooth and hence continuous.

- ii) Show that, for any open $U \subset G$ containing the identity, there exist open $V_1, V_2 \subset U$ containing the identity such that

$$V_1 V_2 := \{gh \mid g \in V_1, h \in V_2\} \subset U.$$

Hint: The map $(g, h) \mapsto gh$ is continuous for the product topology. Be careful to remember how the product topology is defined!

- iii) By taking appropriate intersections, show there exists an open $V \subset U$ containing the identity such that

$$V = V^{-1}, \quad V^2 \subset U.$$

Exercise 7 *Actions of discrete subgroups*

We call a normal subgroup $H \subset G$ of a Lie group *discrete* if there exists an open set $U \ni \text{id}$ such that

$$H \cap U = \{\text{id}\}$$

We say that H acts *properly discontinuously* on G if, around every $g \in G$, there is an open set $U_g \ni g$ such that

$$hU_g \cap U_g = \emptyset \quad \text{for all} \quad \text{id} \neq h \in H, \quad g \in G$$

- i) Show that a subgroup H acts properly discontinuously on G if there exists an open $U \ni \text{id}$ such that

$$hU \cap U = \emptyset \quad \text{for all} \quad \text{id} \neq h \in H$$

- ii) Hence conclude that a discrete normal subgroup acts properly discontinuously on G .

Hint: Consider the open set from the previous problem

Exercise 8 Discrete subgroups are closed

A subgroup $H \subset G$ is called discrete if there exists an open set $U \ni \text{id}$ such that

$$H \cap U = \{\text{id}\}$$

Let us show H contains all its limit points in G and is therefore closed.

- i) Suppose we have a sequence $\{h_n\} \subset H$ such that $\lim_{n \rightarrow \infty} h_n = g \in G$. Show that the sequence

$$x_n = h_n h_{n+1}^{-1}$$

defines a sequence in H such that $\lim_{n \rightarrow \infty} x_n = \text{id}$.

- ii) Hence conclude that $x_n = \text{id}$ for all $n \geq N$ for some n .

Hint: Discreteness and closeness are incompatible

- iii) Hence conclude that $g \in H$.

Exercise 9 Quotient Lie groups

The Quotient Manifold theorem states that, if H is a Lie group acting smoothly on a manifold M , then the orbit space M/H is a manifold if

- H acts freely on M : if $h \cdot x = x$ for any $x \in M$, then $h = \text{id}$.
- H acts properly on M : if K_1 and K_2 are compact subsets of M , then the set

$$\kappa := \{h \in H \mid h \cdot K_1 \cap K_2 \neq \emptyset\}$$

is compact.

A closed normal subgroup $H \subset G$ of a Lie group acts smoothly on G by left multiplication

$$(h, g) \mapsto hg.$$

- i) Show that H acts freely on G .

Hint: No need for Lie theory here

- ii) Show that

$$\kappa = H \cap K_2 K_1^{-1}$$

where

$$K_2 K_1^{-1} = \{g_2 g_1^{-1} \mid g_1 \in K_1, g_2 \in K_2\}$$

- iii) Hence show that a closed subgroup H acts properly on G .

Hint: Why is $K_2 K_1^{-1}$ compact? H is closed, so that would be enough.

- iv) Conclude that G/H is both a group and a manifold for H a closed normal subgroup.

It is in fact a Lie group, but showing the group operations are smooth requires more differential geometry than we have time for