

MAU34103 - Lie algebras and Lie groups

Practice Sheet 1

Trinity College Dublin

Course homepage

These problems are just for practice, to help you warm up for the homework, and get more familiar with the material. I strongly encourage you to give them a try, as the best way to learn maths is through practice. They are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available alongside the problems

Exercise 1 *The special linear Lie algebra*

1. Consider the basis of $\mathfrak{sl}_2(\mathbb{R})$ given by

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Compute the brackets

$$[x, y], [h, x], \text{ and } [h, y]$$

and express your answer in terms of the basis.

2. Hence compute the brackets

$$[x, [x, y]], [h, [x, y]], [2x - 3h, [y + h, x]].$$

3. Determine the (3×3) matrices of the linear operators

$$\text{ad}_x, \text{ad}_y, \text{ad}_h : \mathfrak{sl}_2(\mathbb{R}) \rightarrow \mathfrak{sl}_2(\mathbb{R})$$

in this basis

Exercise 2 *Dimension computations*

Compute the dimension of the following Lie algebras as real vector spaces:

1. $\mathfrak{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$,
2. $\mathfrak{sl}_n(\mathbb{R}) = \{x \in M_n(\mathbb{R}) \mid \text{tr}(x) = 0\}$,
3. $\mathfrak{so}_n(\mathbb{R}) = \{x \in M_n(\mathbb{R}) \mid x + x^T = 0\}$

Exercise 3 *The Heisenberg Lie algebra*

Recall that we defined the Heisenberg Lie algebra by

$$\mathcal{H} = \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}q$$

equipped with Lie bracket

$$[f, q] = 0, \quad [f, g] = \text{coeff}_{z^{-1}}(f'g)q$$

for all $f, g \in \mathbb{C}[z, z^{-1}]$.

1. Verify that this bracket is anticommutative, either showing

$$[f, f] = 0 \quad \text{or} \quad [f, g] + [g, f] = 0.$$

2. Verify that this bracket satisfies the Jacobi identity
3. Show that

$$[z^m, z^n] = m\delta_{m,-n}q$$

where

$$\delta_{a,b} = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

4. Show that $\mathbb{C}[z]$ is a Lie subalgebra, and is, in fact, abelian.

Exercise 4 *A more general Heisenberg algebra*

Given a vector space V , and a bilinear map $\omega : V \times V \rightarrow K$ such that

$$\omega(x, y) + \omega(y, x) = 0$$

we define the Heisenberg Lie algebra

$$\mathcal{H}_\omega := V \oplus Kq$$

with Lie bracket

$$[x, q] = 0, \quad [x, y] = \omega(x, y)q$$

for all $x, y \in V$.

1. Show $\mathcal{H}_\omega$ is a Lie algebra
2. Consider the Heisenberg Lie algebra for $V = \mathbb{C}x \oplus \mathbb{C}y$ with

$$\omega(x, y) = 1.$$

Show that this Lie algebra is isomorphic to the Lie algebra of 3×3 strictly upper triangular matrices:

$$\mathcal{H}_\omega \cong \mathfrak{n}_3.$$

3. Compute the derived Lie algebras $[\mathfrak{sl}_2(\mathbb{C}), \mathfrak{sl}_2(\mathbb{C})]$ and $[\mathfrak{n}_3, \mathfrak{n}_3]$. Hence conclude

$${}_2(\mathbb{C}) \not\cong \mathcal{H}_\omega$$

Exercise 5 *Very Witt-y*

Recall the Witt algebra

$$\text{Der } \mathbb{C}[t] = \mathbb{C}[t] \frac{d}{dt},$$

is a Lie algebra with bracket

$$\left[f \frac{d}{dt}, g \frac{d}{dt} \right] = (fg' - f'g) \frac{d}{dt}.$$

This has a basis

$$L_n = -t^{n+1} \frac{d}{dt}, \quad n \geq -1.$$

Show that

$$[L_m, L_n] = (m - n)L_{m+n}$$

Exercise 6 *Algebraic exponentials*

Let A be an algebra over $\mathbb{R}$, and suppose that $\delta : A \rightarrow A$ is a derivation such that there exists an $N > 0$ such that $\delta^N(x) = 0$ for all $x \in A$.

1. Show that

$$\delta^n(xy) = \sum_{i=0}^n \binom{n}{i} \delta^i(x) \delta^{n-i}(y)$$

for all $n \geq 0$.

2. Hence show that the map

$$\begin{aligned} \exp(\delta) : A &\rightarrow A \\ x &\mapsto \sum_{n \geq 0} \frac{\delta^n(x)}{n!} \end{aligned}$$

is an algebra homomorphism:

$$\exp(\delta)(xy) = \exp(\delta)(x) \exp(\delta)(y)$$

for all $x, y \in A$.

3. Conclude that $\exp(\delta)$ is an isomorphism.

Note: If A is a normed algebra, then we can replace the requirement that $\delta^N(x) = 0$ with $\|\delta(x)\| \leq C\|x\|$ for some constant C , so that the series converges.

Exercise 7 *Two's an anomaly for orthogonal matrices*

Let V be an n -dimensional vector space over a field K , and let $\sigma : V \times V \rightarrow K$ be a non-degenerate bilinear form. By choosing an isomorphism

$$V \cong K^n$$

we can find an invertible $S \in M_n(K)$ such that

$$\sigma(v, w) = v^T S w \text{ for all } v, w \in V$$

1. Show that

$$\mathfrak{o}(V, \sigma) := \{x \in \text{End}(V) \mid \sigma(xv, w) = \sigma(v, xw) \text{ for all } v, w \in V\}$$

is isomorphic to

$$\mathfrak{o}(n, S) = \{x \in M_n(K) \mid x^T S + Sx = 0\}$$

2. Show that, if K has characteristic not equal to two, then

$$x \in \mathfrak{o}(n, S) \Rightarrow \text{tr}(x) = 0$$

3. Now suppose that K has characteristic 2, and consider the case of $S = I$. Determine a basis of

$$\mathfrak{o}(2) = \{x \in M_2(K) \mid x + x^T = 0\}$$

and of

$$\mathfrak{so}(2) = \{x \in \mathfrak{o}(2) \mid \text{tr}(x) = 0\}.$$

Conclude that

$$\mathfrak{so}(2) \subset \mathfrak{o}(2)$$

is a strict inclusion.