

MAU34103 - Lie algebras and Lie groups

Exercise Sheet 2

Trinity College Dublin

Course homepage

Answers are due for October 16th, 23:59.

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *An exponential Lie group (40pts)*

Let

$$N_3 := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

be the Lie group of unipotent upper triangular matrices, and let $\mathfrak{n}_3$ be its Lie algebra

i) (10pts) Show that the series

$$\log(I + X) = - \sum_{n \geq 1} \frac{(-X)^n}{n}$$

is a finite sum for any $I + X \in N_3$.

ii) (15pts) Deduce that $\exp : \mathfrak{n}_3 \rightarrow N_3$ is a bijection, and hence show that $\mathfrak{n}_3$ is the Lie algebra of strictly upper triangular matrices.

iii) (15pts) Show that a Lie group homomorphism $f : N_3 \rightarrow \mathrm{GL}_n(\mathbb{R})$ is injective if and only if the differential $df : \mathfrak{n}_3 \rightarrow \mathfrak{gl}_n(\mathbb{R})$ is injective.

Hint: You may freely use a consequence of Engel's theorem, that $x^m = 0$ for some $m > 0$ implies $df(x)^M = 0$ for some $M > 0$

Solution 1

1. If $I + X \in N_3$, then X is a strictly upper triangular (3×3) -matrix, and so $X^3 = 0$. Thus

$$\log(I + X) = X - \frac{X^2}{2}$$

is a finite sum.

2. As the exponential and logarithm maps are inverse as formal series, we have

$$\exp(\log(X)) = \log(\exp(X)) = X$$

whenever the expressions are defined. As both functions will be given by finite sums, this means that $\exp : \mathfrak{n}_3 \rightarrow N_3$ is a bijection. Thus

$$\mathfrak{n}_3 = \log(N_3)$$

which is clearly a subspace of strictly upper triangular matrices: if $I + X \in N_3$, then X and X^2 are strictly upper triangular. To see that we have equality, we note that $\exp(X) \in N_3$ for all strictly upper triangular X .

While not necessary for the question, it may be helpful to see explicitly that $\exp$ and $\log$ are globally inverse in this case:

$$\begin{aligned} \exp(\log(I + X)) &= \exp(X - X^2/2) \\ &= I + (X - X^2/2) + (X - X^2/2)^2/2 \\ &= I + X - X^2/2 + X^2/2 = I + X \end{aligned}$$

as $X^3 = 0$. Similarly

$$\log(\exp(X)) = \log(I + X + X^2/2) = X + X^2/2 - (X + X^2/2)^2/2 = X.$$

3. If f is injective, a corollary from class tells us df is injective. Conversely, suppose df is injective, but f is not. Then there is some non-identity $X \in N_3$ such that $f(X) = I$. As the exponential map is a bijection here, $X = \exp(x)$ for some non-zero $x \in \mathfrak{n}_3$. Hence

$$I = f(X) = f(e^x) = e^{df(x)}$$

As $df(x)^M = 0$ for some M , we can take the logarithm of both sides to conclude $df(x) = 0$. Thus, if f is not injective, df is not injective.

Important Note!

The original hint for this question implied that $M = m$. As this was my error, students will not be penalised for saying $df(x)^3 = 0$, nor does this meaningfully affect the argument. There is not necessarily a relationship between m and M , but we have the following: if $x^m = 0$ (as a matrix), then we showed that $\text{ad}_x^{2m} = 0$ and hence by the converse to Engel's theorem, the Lie algebra is nilpotent. Thus, the representation is nilpotent.

Exercise 2 *A finite dimensional non-matrix Lie group (60pts)*

A corollary of the Quotient Manifold Theorem states that if $H \subset G$ is a closed normal subgroup of a Lie group, then the quotient group G/H also has the structure of a Lie group of dimension

$$\dim(G/H) = \dim(G) - \dim(H).$$

We will use this to construct an example of a Lie group that is neither a matrix group, nor an abstract universal cover.

You may freely use that discrete subgroups are closed of dimension 0.

- i) (20pts) Let $\pi : G \rightarrow G/H$ be the projection map and let

$$d\pi : \mathfrak{g} \rightarrow L(G/H)$$

be the differential between the corresponding Lie algebras. Show that

$$\ker(d\pi) = \mathfrak{h} = L(H) \subset \mathfrak{g}.$$

- ii) (15pts) Hence show that $d\pi$ is surjective and that

$$L(G/H) \cong \mathfrak{g}/\mathfrak{h}$$

Hint: The dimension of a Lie algebra is equal to the dimension of the Lie group. Compare dimensions.

- iii) (5pts) Hence conclude that, for H a discrete normal subgroup, $L(G/H) \cong \mathfrak{g}$

Hint: Think dimensions!

- iv) (20pts) Let N_3 be the Lie group of unipotent upper triangular matrices as in Exercise 1. There is a discrete normal subgroup

$$Z_3 := \left\{ \begin{pmatrix} 1 & 0 & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$$

As discrete subgroups are closed, the quotient is a Lie group.

Show that there cannot exist an injective Lie group homomorphism $f : N_3/Z_3 \rightarrow \mathrm{GL}_n(\mathbb{R})$ for any $n \geq 0$.

Hint: The differential df must be injective. The map f can be lifted to a map $\tilde{f} : N_3 \rightarrow \mathrm{GL}_n(\mathbb{R})$. Can we compare df and $d\tilde{f}$?

Remark. A subgroup $H \subset G$ is called discrete if there exists an open set $U \ni \mathrm{id}$ such that

$$H \cap U = \{\mathrm{id}\}$$

Solution 2

1. Suppose $d\pi(x) = 0$. Then

$$I = e^{td\pi(x)} = e^{d\pi(tx)} = \pi(e^tx)$$

for all $t \in \mathbb{R}$. Hence $e^{tx} \in H$ for all $t \in \mathbb{R}$ and so $x \in \mathfrak{h}$. Conversely, if $x \in \mathfrak{h}$, then

$$I = \pi(e^{tx}) = e^{td\pi(x)}$$

for all $t \in \mathbb{R}$. Taking the derivative at $t = 0$, we find $d\pi(x) = 0$. Thus $\ker d\pi = \mathfrak{h}$.

2. From the rank-nullity theorem, we have

$$\dim d\pi(\mathfrak{g}) = \dim \mathfrak{g} - \dim \mathfrak{h} = \dim(G) - \dim H = \dim(G/H) = \dim L(G/H).$$

Hence $d\pi(\mathfrak{g}) = L(G/H)$. By the first isomorphism theorem

$$\mathfrak{g} \cong \mathfrak{h} \cong d\pi(\mathfrak{g}) = L(G/H).$$

3. Discrete subgroups have dimension 0, so $\dim \mathfrak{h} = 0$. Therefore $\mathfrak{h} = 0$ and so

$$L(G/H) \cong \mathfrak{g}/\mathfrak{h} = \mathfrak{g}.$$

4. Suppose there exists an injective $f : N_3/Z_3 \rightarrow \mathrm{GL}_n(\mathbb{R})$. Then we have an injective map

$$df : \mathfrak{g} \rightarrow \mathfrak{gl}_n(\mathbb{R})$$

where $\mathfrak{g} = L(N_3/Z_3)$. We can also lift f to a map

$$\begin{aligned} \tilde{f} : N_3 &\rightarrow \mathrm{GL}_n(\mathbb{R}) \\ X &\mapsto f(\pi(X)). \end{aligned}$$

This satisfies $\pi \circ \tilde{f} = f$ and so, by chain rule,

$$d\pi \circ d\tilde{f} = df.$$

As Z_3 is discrete, $d\pi$ is an isomorphism. Hence $d\tilde{f}$ is injective if and only if df is injective. But, by Exercise 1, $d\tilde{f}$ is injective if and only if $\tilde{f}$ is injective. But $Z_3 \subset \ker \tilde{f}$, so $\tilde{f}$ is not injective. Thus, no injective map $f : N_3/Z_3 \rightarrow \mathrm{GL}_n(\mathbb{R})$ exists.

These are the only exercises that you must submit before the deadline

Further exercises on this topic can be found on the course webpage, and, I strongly encourage you to give them a try, as the best way to learn maths is through practice.

They are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available alongside the problems

Optional: A sketch of Lie's second theorem

For anyone who is interested, the following is a series of exercises intended to lead you to a proof that every Lie algebra homomorphism induces a Lie group homomorphism, under the assumption that the domain is connected and simply connected. We will also impose the additional assumption that we work with linear groups.

Let G be a linear group with Lie algebra $\mathfrak{g}$, and let $\exp : \mathfrak{g} \rightarrow G$ be the standard exponential. We know there is an open neighbourhood $U \subset \mathfrak{g}$ of 0 and an open neighbourhood $V \subset G$ of I such that $\exp : U \rightarrow V$ is a local homeomorphism.

- i) Suppose that G is (path) connected and let $g \in G$. Suppose $\gamma : [0, 1] \rightarrow G$ is a continuous path between I and g . Show that there exists a sequence

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$$

such that

$$\gamma(t_k)\gamma(t_{k-1})^{-1} \in V$$

for all $1 \leq k \leq n$.

- ii) Hence conclude there exists $x_1, \dots, x_n \in U \subset \mathfrak{g}$ such that

$$g = e^{x_n} e^{x_{n-1}} \cdots e^{x_2} e^{x_1}$$

As such, conclude that G is the subgroup of $\mathrm{GL}_n(\mathbb{R})$ generated by $\exp(\mathfrak{g})$.

- iii) Given a Lie algebra homomorphism $f : \mathfrak{g} \rightarrow \mathfrak{h}$, where $\mathfrak{h}$ is the Lie algebra of a linear group H , we can define a map $F : V \rightarrow H$ by

$$F(e^x) := e^{f(x)}$$

Show that if $e^x e^y = e^z$, with $x, y, z \in U$, then

$$F(e^x)F(e^y) = F(e^z)$$

Hint: Assume that the Baker-Campbell-Hausdorff formula holds on U .

- iv) We would like to extend this to a homomorphism on all of G by

$$F(e^{x_n} \cdots e^{x_1}) := F(e^{x_n})F(e^{x_{n-1}}) \cdots F(e^{x_1})$$

however, we do not have that there is a unique way to write

$$g = e^{x_n} e^{x_{n-1}} \cdots e^{x_2} e^{x_1}$$

as so this could be poorly defined.

Show that, if we fix a path γ , defining $F(g)$ this way is independent of choice our partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$$

satisfying the conditions of part (i).

- v) Finally show that, given two paths γ_1 and γ_2 between I and g , and associated partitions, defining $F(g)$ in this way is independent of our choice of path

Hint: As G is simply connected, we have a continuous set of paths $H(s, t)$ such that $H(0, t) = \gamma_1(t)$ and $H(1, t) = \gamma_2(t)$. As such, we can partition this set of paths into paths that are “close enough”. By taking an appropriately small partition of each path, write each e^{z_i} in the form $e^{x_i} e^{y_i}$ for $y_i \in U$.