

MAU34103 - Lie algebras and Lie groups

Exercise Sheet 1

Trinity College Dublin

Course homepage

Answers are due for October 2th, 23:59.

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Isomorphism classes of real Lie algebras (40pts)*

Let us try to describe all low-dimensional Lie algebras over $\mathbb{R}$ up to isomorphism.

1. (10pts) We say two Lie algebras are isomorphic if there is a Lie algebra homomorphism

$$f : \mathfrak{g} \rightarrow \mathfrak{h}$$

such that f is a bijection. Show that the inverse function

$$f^{-1} : \mathfrak{h} \rightarrow \mathfrak{g}$$

is also a Lie algebra homomorphism.

2. (10pts) Describe all real Lie algebras of dimension 1 up to isomorphism.
3. (20pts) Describe all real Lie algebras of dimension 2 up to isomorphism

Hint for both: The Lie bracket is entirely determined by the bracket of basis elements.

Solution 1

1. Let f be an isomorphism with inverse f^{-1} . That f^{-1} is linear is well known, so it remains to check that

$$f^{-1}[u, v] = [f^{-1}(u), f^{-1}(v)]$$

for all $u, v \in \mathfrak{h}$. As f is surjective, there exist $x, y \in \mathfrak{g}$ such that $u = f(x)$, $v = f(y)$, and so we want to show that

$$f^{-1}[f(x), f(y)] = [x, y]$$

As f is a bijection, this is equivalent to

$$[f(x), f(y)] = f([x, y])$$

which is true as f is a homomorphism.

2. It is enough to describe all Lie algebra structures on $\mathbb{R}$. As the bracket is bilinear, it is entirely determined by $[1, 1]$, which is 0 by definition. Therefore, there is a unique Lie algebra structure on $\mathbb{R}$, with the trivial bracket.
3. It is enough to describe all Lie algebra structures, up to isomorphism, on $\mathbb{R}^2$. Let e_1, e_2 be the standard basis. The bracket is determined by

$$[e_1, e_1] = 0, \quad [e_1, e_2] = z, \quad [e_2, e_2] = 0$$

for some $z \in \mathbb{R}^2$. It is easy to check that the Jacobi identity imposes no additional restrictions, so we will omit it (and not penalise students for forgetting it). Thus a Lie algebra structure is determined by a choice of $z = pe_1 + qe_2$.

To determine the Lie algebra structures up to isomorphism, we first note that the abelian Lie algebra with $z = 0$ is not isomorphic to any Lie algebra with $z \neq 0$.

So let us consider $z \neq 0$ and suppose that

$$f : (\mathbb{R}^2, [-, -]_1) \rightarrow (\mathbb{R}^2, [-, -]_2)$$

is an isomorphism. It is therefore a linear map, sending

$$e_1 \mapsto ae_1 + ce_2, \quad e_2 \mapsto be_1 + de_2$$

for some $a, b, c, d \in \mathbb{R}$ such that $ad - bc \neq 0$. Let $z = pe_1 + qe_2$.

As f preserves the Lie bracket, we must have

$$\begin{aligned}(pa + bq)e_1 + (pc + qd)e_2 &= f(z) \\ &= [f(e_1), f(e_2)]_2 = [ae_1 + ce_2, be_1 + de_2]_2 \\ &= (ad - bc)[e_1, e_2]_2 = (ad - bc)(Pe_1 + Qe_2)\end{aligned}$$

Thus, given (p, q) and (P, Q) , the corresponding Lie algebras are isomorphic if and only if we can find an invertible $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\begin{pmatrix} P \\ Q \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} pa + qb \\ pc + qd \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}$$

This is always possible, as $\text{GL}_2(\mathbb{R})$ acts transitively on $\mathbb{R}^2 \setminus \{0\}$, and we can always scale a, b, c, d to take care of the determinant. In fact, if we take $p = 1, q = 0$, then we can see that by taking

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} P & -Q \\ Q & 0 \end{pmatrix}$$

works for all $Q \neq 0$, and a similar matrix can be constructed for $P \neq 0$.

Exercise 2 *Real or complex? It's complicated*

Define the unitary Lie algebra to be the real $2n^2$ -dimensional vector space

$$\mathfrak{u}(n) := \{A \in M_n(\mathbb{C}) \mid A + \overline{A}^T = 0\}$$

where $\overline{A}^T$ is the conjugate transpose of A , equipped with the commutator

$$[A, B] = AB - BA.$$

1. (10pts) Show that $\mathfrak{u}(n)$ is *not* a complex Lie algebra.

Hint: You don't need Lie algebras to do this

2. (15pts) By splitting into real and imaginary parts, determine an injective linear map

$$\iota : M_n(\mathbb{C}) \rightarrow M_{2n}(\mathbb{R})$$

such that $\iota(AB) = \iota(A)\iota(B)$.

Hint: Try doing it for $n = 1$ first, and think in blocks

3. (20pts) Hence show that $\mathfrak{u}(n)$ can be considered as a Lie subalgebra of

$$\mathfrak{so}(2n) = \{M \in M_{2n}(\mathbb{R}) \mid M + M^T = 0\}$$

and determine $\dim_{\mathbb{R}} \mathfrak{u}(n)$.

4. (15pts) Consider the $\mathbb{C}$ -linear map from the tensor product

$$\begin{aligned} c : \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{u}(n) &\rightarrow \mathfrak{gl}_n(\mathbb{C}), \\ z \otimes A &\mapsto zA. \end{aligned}$$

Show this is an isomorphism of *complex* vector spaces.

Hint: Either construct an inverse, or show it is injective/surjective and compare dimensions.

Solution 2

1. If it were a complex Lie algebra, then for any $A \in \mathfrak{u}(n)$, we would also have $iA \in \mathfrak{u}(n)$ and hence

$$A + \overline{A}^T = iA - i\overline{A}^T = 0$$

which implies $A = 0$. But $\mathfrak{u}(n)$ is non-zero. Thus, it cannot be a complex Lie algebra.

2. It is not too hard to show that if $A = P + iQ$, then defining

$$\iota(A) = \begin{pmatrix} P & Q \\ -Q & P \end{pmatrix}$$

satisfies the requirement. It is clearly injective, linear, and we find:

$$\begin{aligned} \iota(AB) &= \iota((P + iQ)(S + iT)) \\ &= \iota((PS - QT) + i(PT + QS)) \\ &= \begin{pmatrix} PS - QT & PT + QS \\ -PT - QS & PS - QT \end{pmatrix} \\ &= \begin{pmatrix} P & Q \\ -Q & P \end{pmatrix} \begin{pmatrix} S & T \\ -T & S \end{pmatrix} \\ &= \iota(A)\iota(B). \end{aligned}$$

3. As $\iota(AB) = \iota(A)\iota(B)$, we have

$$\iota([A, B]) = [\iota(A), \iota(B)]$$

so $\mathfrak{u}(n)$ can be considered as a Lie subalgebra of $\mathfrak{gl}_{2n}(\mathbb{R})$. The condition that

$$A + \overline{A}^T = 0$$

implies (writing out the matrices)

$$\iota(A) + \iota(A)^T = 0$$

and hence $\mathfrak{u}(n) \subset \mathfrak{so}(2n)$.

To determine the dimension of $\mathfrak{u}(n)$, note that if $A = P + iQ$, the condition imposes that $P + P^T = 0$ and $Q - Q^T = 0$. Thus P is antisymmetric, has 0 along the diagonal, and the lower triangle is the negative of the upper triangle. Similarly Q is symmetric, so the lower triangle is equal to the upper triangle, and we can choose the diagonal freely. This

$$\dim_{\mathbb{R}} \mathfrak{u}(n) = \frac{n(n+1)}{2} + \frac{n(n-1)}{2} = n^2.$$

(We get $1+2+\cdots+n$ degrees of freedom from Q and $1+2+\cdots+n-1 = \frac{n(n-1)}{2}$ degrees of freedom from P .)

4. Let us show that it is injective. Suppose that $c(z \otimes A) = 0$. Then $zA = 0$. Clearly, either $z = 0$ or $A = 0$, and as $0 \times A = z \otimes 0 = 0$, we therefore have $z \otimes A = 0$. So the map is injective.

The complex dimension of $\mathfrak{gl}_n(\mathbb{C})$. The complex dimension of $\mathbb{C} \otimes \mathfrak{u}(n)$ is the real dimension of $\mathfrak{u}(n)$, namely n^2 . Thus, they are isomorphic.

Alternatively, one can show that

$$c^{-1}(A + iB) = 1 \otimes iB - i \otimes iA$$

is an inverse.

These are the only exercises that you must submit before the deadline

Further exercises on this topic can be found on the course webpage, and, I strongly encourage you to give them a try, as the best way to learn maths is through practice.

They are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available alongside the problems

A reminder on the tensor product

The tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{u}(n)$, called the *complexification* of $\mathfrak{u}(n)$ gives a way to convert a real vector space into a complex one. For a real vector space V , it is defined as follows.

The complexification of V is the complex vector space $\mathbb{C} \otimes_{\mathbb{R}} V$ spanned by symbols $z \otimes v$, with $z \in \mathbb{C}$ and $v \in V$ subject to

1. $(z_1 + z_2) \otimes v = z_1 \otimes v + z_2 \otimes v$ for all $z_1, z_2 \in \mathbb{C}$, $v \in V$,
2. $z \otimes (v_1 + v_2) = z \otimes v_1 + z \otimes v_2$ for all $z \in \mathbb{C}$, $v_1, v_2 \in V$,
3. $(\lambda z) \otimes v = z \otimes (\lambda v) = \lambda(z \otimes v)$ for all $\lambda \in \mathbb{R}$.

Scalar multiplication is defined by

$$w(z \otimes v) := (wz) \otimes v \text{ for all } w \in \mathbb{C}.$$

This equips $\mathbb{C} \otimes_{\mathbb{R}} V$ with the structure of a complex vector space of dimension $\dim_{\mathbb{R}} V$.

Remark. *The subscript $\mathbb{R}$ serves to tell you what scalars can “cross” the tensor product $\otimes$, and is often omitted when it is clear from context.*