

# MAU34103 - Lie algebras and Lie groups

Lie's Second Theorem

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Course homepage

## A sketch of Lie's second theorem

For anyone who is interested, the following are the solutions to the series of exercises intended to lead you to a proof that every Lie algebra homomorphism induces a Lie group homomorphism, under the assumption that the domain is connected and simply connected. We will also impose the additional assumption that we work with linear groups.

Let  $G$  be a linear group with Lie algebra  $\mathfrak{g}$ , and let  $\exp : \mathfrak{g} \rightarrow G$  be the standard exponential. We know there is an open neighbourhood  $U \subset \mathfrak{g}$  of 0 and an open neighbourhood  $V \subset G$  of  $I$  such that  $\exp : U \rightarrow V$  is a local homeomorphism.

- i) Suppose that  $G$  is (path) connected and let  $g \in G$ . Suppose  $\gamma : [0, 1] \rightarrow G$  is a continuous path between  $I$  and  $g$ . Show that there exists a sequence

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$$

such that

$$\gamma(t_k)\gamma(t_{k-1})^{-1} \in V$$

for all  $1 \leq k \leq n$ .

- ii) Hence conclude there exists  $x_1, \dots, x_n \in U \subset \mathfrak{g}$  such that

$$g = e^{x_n} e^{x_{n-1}} \cdots e^{x_2} e^{x_1}$$

As such, conclude that  $G$  is the subgroup of  $\mathrm{GL}_n(\mathbb{R})$  generated by  $\exp(\mathfrak{g})$ .

- iii) Given a Lie algebra homomorphism  $f : \mathfrak{g} \rightarrow \mathfrak{h}$ , where  $\mathfrak{h}$  is the Lie algebra of a linear group  $H$ , we can define a map  $F : V \rightarrow H$  by

$$F(e^x) := e^{f(x)}$$

Show that if  $e^x e^y = e^z$ , with  $x, y, z \in U$ , then

$$F(e^x)F(e^y) = F(e^z)$$

*Hint: Assume that the Baker-Campbell-Hausdorff formula holds on  $U$ .*

- iv) We would like to extend this to a homomorphism on all of  $G$  by

$$F(e^{x_n} \cdots e^{x_1}) := F(e^{x_n})F(e^{x_{n-1}}) \cdots F(e^{x_1})$$

however, we do not have that there is a unique way to write

$$g = e^{x_n} e^{x_{n-1}} \cdots e^{x_2} e^{x_1}$$

as so this could be poorly defined.

Show that, if we fix a path  $\gamma$ , defining  $F(g)$  this way is independent of choice our partition

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1$$

satisfying the conditions of part (i).

- v) Finally show that, given two paths  $\gamma_1$  and  $\gamma_2$  between  $I$  and  $g$ , and associated partitions, defining  $F(g)$  in this way is independent of our choice of path

*Hint: As  $G$  is simply connected, we have a continuous set of paths  $H(s, t)$  such that  $H(0, t) = \gamma_1(t)$  and  $H(1, t) = \gamma_2(t)$ . As such, we can partition this set of paths into paths that are “close enough”. By taking an appropriately small partition of each path, write each  $e^{z_i}$  in the form  $e^{x_i} e^{y_i}$  for  $y_i \in U$ .*

### Sketchy details of the argument

- i) Since the set  $[0, 1]$  is compact, the continuous map  $\gamma : [0, 1] \rightarrow G$  is uniformly continuous. Hence, for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that  $|s - t| < \delta$  implies  $\|\gamma(s) - \gamma(t)\| < \varepsilon$ .

Furthermore, as  $t \mapsto \|\gamma(t)^{-1}\|$  is continuous, it is bounded by some  $C > 0$ . Thus, for any  $\varepsilon > 0$ , we can find  $\delta > 0$  such that  $|s - t| < \delta$  implies

$$\|\gamma(s)\gamma(t)^{-1} - I\| \leq \|\gamma(s) - \gamma(t)\| \|\gamma(t)^{-1}\| < \frac{\varepsilon}{C} C = \varepsilon.$$

From the existence of the local homeomorphism, we have an open set  $V$  containing  $I$  such that every element of  $V$  is an exponential of an element of  $\mathfrak{g}$ . In particular, we can find an open ball  $B_G(I, \eta)$  such that every element in the open ball is an exponential.

Taking this  $\eta$  as our  $\varepsilon$ , we therefore have a  $\delta > 0$  such that  $|s - t| < \delta$  implies that  $\gamma(s)\gamma(t)^{-1} \in B_G(I, \eta)$  and hence  $\gamma(s)\gamma(t) = e^{x_{s,t}}$  for some  $x_{s,t} \in \mathfrak{g}$ . Then, taking any sequence

$$0 = t_0 < t_1 < \dots < t_n = 1$$

such that  $t_{i+1} - t_i < \delta$  gives the desired result.

- ii) Taking  $t_1, \dots, t_{n-1}$  as in the first part, we can find  $x_i$  such that  $\gamma(t_1) = e^{x_1}$ ,  $\gamma(t_2) = e^{x_2}\gamma(t_1) = e^{x_2}e^{x_1}$ , and iterate this to find

$$g = \gamma(1) = e^{x_n}\gamma(t_{n-1}) = e^{x_n}e^{x_{n-1}} \dots e^{x_1}.$$

Since every product of exponentials will be an element of  $G$  and every element of  $G$  can be expressed as a product of exponentials in this way, we have that  $G$  is the subgroup of  $\text{GL}_n(\mathbb{R})$  generated by  $\exp(\mathfrak{g})$ .

- iii) Assume the Baker-Campbell-Hausdorff formula hold and let  $BCH(x, y)$  be the element defined by the series. Then, as  $f$  is a Lie algebra homomorphism, and  $BCH(x, y)$  is a series in Lie brackets, we must have

$$BCH(f(x), f(y)) = f(BCH(x, y))$$

Thus

$$F(e^x)F(e^y) = e^{f(x)}e^{f(y)} = e^{BCH(f(x), f(y))} = e^{f(BCH(x, y))} = F(e^z) = F(e^xe^y)$$

*The assumptions for this part aren't quite right. We actually need to restrict to a small enough set such that BCH series converges on both sides. We can do this as  $f$  is continuous, so this isn't too onerous.*

iv) Suppose we have two partitions of  $[0, 1]$

$$0 < t_1 < \dots < t_{n-1} < 1, \quad 0 < s_1 < \dots < s_{m-1} < 1$$

with corresponding elements of  $\mathfrak{g}$   $x_1, \dots, x_n$  and  $y_1, \dots, y_m$ . We can take a common refinement of these partitions

$$0 < q_1 < q_2 < \dots < q_{N-1} < 1$$

with corresponding elements  $z_1, \dots, z_N$ . We have

$$e^{x_n} \dots e^{x_1} = e^{z_N} \dots e^{z_1} = e^{y_m} \dots e^{y_1}.$$

As we have refined our partitions, we must be able to factorise this and write each  $e^{x_i}$  and  $e^{y_j}$  as a product of  $e^{z_k}$ :

$$e^{x_i} = e^{z_b} e^{z_{b-1}} \dots e^{z_a}, \quad e^{y_j} = e^{z_d} e^{z_{d-1}} \dots e^{z_c}.$$

From the previous part, we will have that

$$F(e^{x_i}) = F(e^{z_b}) \dots F(e^{z_a}), \quad F(e^{y_j}) = F(e^{z_d}) \dots F(e^{z_c})$$

and hence

$$F(e^{x_n}) \dots F(e^{x_1}) = F(e^{z_N}) \dots F(e^{z_1}) = F(e^{y_m}) \dots F(e^{y_1})$$

Thus defining  $F(g) = F(e^{x_n}) \dots F(e^{x_1})$  is independent of our choice of partition.

v) Analogously to earlier, continuity of  $H$  and compactness of  $[0, 1]^2$  implies we can find a partition

$$0 = s_0 < s_1 < \dots < s_m = 1, \quad 0 < t_1 < \dots < t_n = 1$$

of  $[0, 1] \times [0, 1]$  such that

$$H(s_i, t_j)H(s_i, t_{j-1})^{-1} = e^{x_{i,j}}, \quad H(s_i, t_j)H(s_{i-1}, t_j)^{-1} = e^{y_{i,j}}$$

where  $y_{i,n} = y_{i,0} = 0$ .

We can furthermore assume the partition is fine enough that

$$F(e^{y_{i,j}})F(e^{x_{i-1,j}}) = F(e^{x_{i,j}})F(e^{y_{i,j-1}})$$

Hence

$$\begin{aligned} & F(e^{x_{i-1,n}})F(e^{x_{i-1,n-1}}) \dots F(e^{x_{i-1,1}}) \\ &= F(e^{x_{i,n}})F(e^{y_{i,n-1}})F(e^{y_{i,n-1}})^{-1}F(e^{x_{i,n-1}})F(e^{y_{i,n-2}}) \dots F(e^{y_{i,1}})^{-1}F(e^{x_{i,1}}) \\ &= F(e^{x_{i,n}})F(e^{x_{i,n-1}}) \dots F(e^{x_{i,1}}) \end{aligned}$$

Hence, defining  $F(g)$  via  $H(0, t)$  or  $H(1, t)$  produces the same result. Thus  $F(g)$  is well defined.