MAU34103 - Lie algebras and Lie groups

Exercise Sheet 4

Trinity College Dublin

Course homepage

Answers are due for November 20th, 23:59.
The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Decomposing representations (40pts)*

In all that follows, let $V(1)$ denote the 2-dimensional irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$.

i) (10pts) Determine the h-eigenvectors of the $\mathfrak{sl}_2$-representation $V(1) \otimes V(1)$.

ii) (15pts) Hence determine $\{k_m\}_{m\geq 0}$ such that

$$V(1) \otimes V(1) \cong \bigoplus_{m\geq 0} V(m)^{\oplus k_m}$$

and give a basis of h-eigenvectors compatible with this decomposition. That is, give a basis for each component $V(n)$.

iii) (10pts) Determine the matrices of the actions of x and y on $V(1)\otimes V(1)$ in the basis of h-eigenvectors

Hint: How do x and y act in the direct sum decomposition?

iv) (5pts) Show that the dual representation $V(1)^*$ is isomorphic to $V(1)$.

Hint: What are the possible two dimensional representations?

Exercise 2 *For the induction haters (60pts)*

An extremely common exercise to introduce proof by induction is to show the following

$$1 + 3 + 5 + \cdot + 2n + 1 = (n + 1)^2.$$

The goal of this exercise is to give the worst possible proof of this fact.

i) (15pts) Let $V(n)$ be irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ of dimension $n + 1$, and let

$$W(n) = \mathfrak{gl}(V(n)).$$

We define a map $\rho : \mathfrak{sl}_2 \to W(n)$ by

$$\rho(z)(A)(v) := z(Av) - A(zv)$$

for all $z \in \mathfrak{sl}_2$, $A \in \mathfrak{gl}(V(n))$, $v \in V(n)$. Show that $(W(n), \rho)$ defines an $\mathfrak{sl}_2$-representation.

ii) (15pts) Determine a basis of eigenvectors of $rho(h)$.

Hint: Fix a good basis of $V(n)$ and try the obvious thing

iii) (10pts) Determine the dimensions of the eigenspaces

$$W(n)_m = \{A \in W(n) \mid \rho(h)A = mA\}$$

iv) (10pts) Determine $\{k_m\}_{m \geq 0}$ such that

$$W(n) \cong \bigoplus_{m \geq 0} V(m)^{\oplus k_m}$$

v) (10pts) Hence conclude

$$(n + 1)^2 = \sum_{m=0}^{n} 2m + 1$$

Hint: Compare dimensions!

Remark. *This is one of many examples of using representation theory to overcomplicate things! Another example will appear in Practice Sheet 4, where we will use $\mathfrak{sl}_2$-representations to prove*

$$\binom{n}{0} < \binom{n}{1} < \cdots < \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil} > \cdots > \binom{n}{n-1} > \binom{n}{n}.$$

by defining an $\mathfrak{sl}_2$ action on a vector space with basis vectors indexed by simple graphs.

These are the only exercises that you must submit before the deadline

Further exercises on this topic can be found on the course webpage, and, I strongly encourage you to give them a try, as the best way to learn maths is through practice.

They are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available alongside the problems.