

MAU34103 - Lie algebras and Lie groups

Exercise Sheet 3

Trinity College Dublin

Course homepage

Answers are due for November 6th, 23:59.

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Representations matter! (30pts)*

- i) (10pts) Show that, for any complex L representation (V, ρ) , (V^*, ρ^*) is a representation of L , where $V^* = \text{Hom}(V, \mathbb{C})$ and

$$\rho^*(x)(f)(v) := -f(\rho(x)v).$$

- ii) (10pts) Show that, for L finite dimensional, σ a non-degenerate, L -invariant bilinear form, the map

$$\begin{aligned} L &\mapsto L^* \\ x &\mapsto \sigma(x, -) \end{aligned}$$

is an isomorphism of representations

$$(L, \text{ad}) \cong (L^*, \text{ad}^*).$$

- iii) (10pts) Show that, if L is a simple Lie algebra, then any non-degenerate, L -invariant bilinear form is a scalar multiple of the Killing form.

Hint: Schur's Lemma! Can you construct an isomorphism from $(L, \text{ad}) \rightarrow (L, \text{ad})$ that would be relevant?

Exercise 2 *Nilpotent nihilism (35pts)*

Recall the Heisenberg Lie algebra $\mathcal{H}$, which has basis x, y, q and Lie brackets

$$[x, y] = q, [x, q] = [y, q] = 0$$

- i) (10pts) Show that the central series of $\mathcal{H}$ is finite and hence $\mathcal{H}$ is nilpotent.
- ii) (10pts) Show that $\mathbb{C}^3$ is a representation of $\mathcal{H}$ via the map

$$\rho(x) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \rho(y) = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \rho(q) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

- iii) (15pts) Engel's theorem tells us that there exists a basis in which $\rho(x), \rho(y), \rho(q)$ are strictly upper triangular. Find one.

Exercise 3 *Practice with Killing (35pts)*

- i) (10pts) Let $\mathfrak{sl}_2(\mathbb{C})$ be the 3-dimensional complex Lie algebra with basis x, y, h and Lie bracket

$$[x, y] = h, [h, x] = 2x, [h, y] = -2y$$

and consider the adjoint representation

$$\begin{aligned} \text{ad} : \mathfrak{sl}_2(\mathbb{C}) &\rightarrow \mathfrak{gl}(\mathfrak{sl}_2(\mathbb{C})) \\ z &\mapsto \text{ad}_z. \end{aligned}$$

Determine the matrices of $\text{ad}_x, \text{ad}_y, \text{ad}_h$ in the basis x, y, z .

Recall: these will be 3×3 matrices!

- ii) (15pts) Determine the Killing form for $\mathfrak{sl}_2(\mathbb{C})$ by computing the nine possible values of

$$\kappa(a, b), \quad a, b \in \{x, y, h\}.$$

Hence show that κ is non-degenerate and conclude $\mathfrak{sl}_2(\mathbb{C})$ is semisimple.

- iii) (10pts) Determine the dual basis to x, y, h with respect to κ .

These are the only exercises that you must submit before the deadline

Further exercises on this topic can be found on the course webpage, and, I strongly encourage you to give them a try, as the best way to learn maths is through practice.

They are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available alongside the problems.