

MAU34103 - Lie algebras and Lie groups

Exercise Sheet 1

Trinity College Dublin

Course homepage

Answers are due for October 2th, 23:59.

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Isomorphism classes of real Lie algebras (40pts)*

Let us try to describe all low-dimensional Lie algebras over $\mathbb{R}$ up to isomorphism.

1. (10pts) We say two Lie algebras are isomorphic if there is a Lie algebra homomorphism

$$f : \mathfrak{g} \rightarrow \mathfrak{h}$$

such that f is a bijection. Show that the inverse function

$$f^{-1} : \mathfrak{h} \rightarrow \mathfrak{g}$$

is also a Lie algebra homomorphism.

2. (10pts) Describe all real Lie algebras of dimension 1 up to isomorphism.
3. (20pts) Describe all real Lie algebras of dimension 2 up to isomorphism

Hint for both: The Lie bracket is entirely determined by the bracket of basis elements.

Exercise 2 *Real or complex? It's complicated*

Define the unitary Lie algebra to be the real $2n^2$ -dimensional vector space

$$\mathfrak{u}(n) := \{A \in M_n(\mathbb{C}) \mid A + \overline{A}^T = 0\}$$

where $\overline{A}^T$ is the conjugate transpose of A , equipped with the commutator

$$[A, B] = AB - BA.$$

1. (10pts) Show that $\mathfrak{u}(n)$ is *not* a complex Lie algebra.

Hint: You don't need Lie algebras to do this

2. (15pts) By splitting into real and imaginary parts, determine an injective linear map

$$\iota : M_n(\mathbb{C}) \rightarrow M_{2n}(\mathbb{R})$$

such that $\iota(AB) = \iota(A)\iota(B)$.

Hint: Try doing it for $n = 1$ first, and think in blocks

3. (20pts) Hence show that $\mathfrak{u}(n)$ can be considered as a Lie subalgebra of

$$\mathfrak{so}(2n) = \{M \in M_{2n}(\mathbb{R}) \mid M + M^T = 0\}$$

and determine $\dim_{\mathbb{R}} \mathfrak{u}(n)$.

4. (15pts) Consider the $\mathbb{C}$ -linear map from the tensor product

$$\begin{aligned} c : \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{u}(n) &\rightarrow \mathfrak{gl}_n(\mathbb{C}), \\ z \otimes A &\mapsto zA. \end{aligned}$$

Show this is an isomorphism of *complex* vector spaces.

Hint: Either construct an inverse, or show it is injective/surjective and compare dimensions.

These are the only exercises that you must submit before the deadline

Further exercises on this topic can be found on the course webpage, and, I strongly encourage you to give them a try, as the best way to learn maths is through practice.

They are arranged by theme, and roughly in order of difficulty within each theme, with the first few working as good warm-ups, and the remainder being of similar difficulty to the main exercise. You are welcome to email me if you have any questions about them. The solutions will be made available alongside the problems

A reminder on the tensor product

The tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{u}(n)$, called the *complexification* of $\mathfrak{u}(n)$ gives a way to convert a real vector space into a complex one. For a real vector space V , it is defined as follows.

The complexification of V is the complex vector space $\mathbb{C} \otimes_{\mathbb{R}} V$ spanned by symbols $z \otimes v$, with $z \in \mathbb{C}$ and $v \in V$ subject to

1. $(z_1 + z_2) \otimes v = z_1 \otimes v + z_2 \otimes v$ for all $z_1, z_2 \in \mathbb{C}$, $v \in V$,
2. $z \otimes (v_1 + v_2) = z \otimes v_1 + z \otimes v_2$ for all $z \in \mathbb{C}$, $v_1, v_2 \in V$,
3. $(\lambda z) \otimes v = z \otimes (\lambda v) = \lambda(z \otimes v)$ for all $\lambda \in \mathbb{R}$.

Scalar multiplication is defined by

$$w(z \otimes v) := (wz) \otimes v \text{ for all } w \in \mathbb{C}.$$

This equips $\mathbb{C} \otimes_{\mathbb{R}} V$ with the structure of a complex vector space of dimension $\dim_{\mathbb{R}} V$.

Remark. *The subscript $\mathbb{R}$ serves to tell you what scalars can “cross” the tensor product $\otimes$, and is often omitted when it is clear from context.*