Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Science, Technology, Engineering and Mathematics

School of Mathematics

Sample Exam 2 **Hilary Term**

Euclidean and non-Euclidean Geometry

Day Place Time

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Instructions to candidates:

Attempt any three questions from the appropriate section. Undergraduates should attempt only Section A. Students taking the PME should attempt only Section B.

If you attempt all four questions, only your best three will be considered in your grade. Each question is worth 33 points. One point will be awarded for general presentation.

Unless stated otherwise, you may use all statements given in lectures without proof, but must clearly justify that the assumptions of statement are fulfilled.

Additional instructions for this examination:

Formula and tables are available from the invigilators if required. A list of Euclidean Propositions is provided on the final page

You may use a non-programmable calculator. Please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

Section A - Undergraduate

Question 1

i) (10 pts) Define a (Euclidean) polygon.

ii) (8 pts) Describe the group of isometries of the Euclidean plane.

iii) (8 pts) Give a formula for the area of a hyperbolic triangle in terms of its angles.

iv) (7 pts) Define the Gaussian curvature of a space at a point.

Question 2

i) (15 pts) Consider the metric space $(\mathbb{R}^2, d_E)$ with the Euclidean metric. Let $\alpha, \beta, \gamma \in \mathbb{R}_{>0}$ be constant, with $\alpha \neq \beta$. Show that there exists $x_1, x_2, C \in \mathbb{R}$ such that the sets

$$E := \{(x, y) \in \mathbb{R}^2 \mid \alpha x^2 + \beta y^2 = \gamma\}$$

and

$$\{(x, y) \in \mathbb{R}^2 \mid d\left((x, y), (x_1, 0)\right) + d\left((x, y), (x_2, 0)\right) = C$$

coincide.

ii) (15 pts) Let $A = (x_1, 0)$, $B = (x_2, 0)$, and $C \in E$. Let ℓ_0 be the bisector of the angle $\angle ACB$. Show that the perpendicular line to ℓ_0 through C intersects E only at C

iii) (3 pts) Does this hold true if $\alpha = \beta$?

Question 3

Fix the lattice $\Lambda = \mathbb{Z}^2 \subset \mathbb{R}^2$. We will give an alternative proof that all elementary triangles have area $\frac{1}{2}$ to the one in class.

i) (10 pts) Define a lattice polygon and an elementary triangle.

ii) (10 pts) Let T be an elementary triangle. Show that there exists a lattice rectangle R with sides parallel to the axes of minimal area containing T, and give the coordinates of its vertices in terms of those of T.

iii) (13 pts) Compute the ratio $\text{Area}(T)/\,\text{Area}(R)$ and hence that $\text{Area}(T) = \frac{1}{2}$

Question 4

i) (10 pts) Define a projective transformation of real projective plane and show that these preserve the cross ratio

$$(x, y; z, w) := \frac{(z-x)(w-y)}{z-y)(w-x)}$$

of 4 collinear points x, y, z, w.

ii) (10 pts) Show that three points $[a_1 : a_2 : a_3], [b_1 : b_2 : b_3], [c_1 : c_2 : c_3]$ are collinear in $\mathbb{RP}^2$ if and only if

$$\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = 0$$

iii) (13 pts) Hence show Pappus's theorem: if A, B, C are collinear, and a, b, c are collinear, then the points

$$X = Ab \cap aB,\ Y = Ac \cap aC,\ Z = Bc \cap bC$$

are also collinear.

Hint: Using a projective transformation, you may assume that $C = [1 : 0 : 0]$, $c = [0 : 1 : 0]$, $X = [0 : 0 : 1]$, *and* $A = [1 : 1 : 1]$

Section B - Postgraduate

Question 1

i) (10 pts) Define a (Euclidean) polygon.

ii) (8 pts) Define an affine transformation of the Euclidean plane.

iii) (8 pts) Define an inner product in a vector space.

iv) (7 pts) Give three examples of spherical projection, including their domains of definition.

Question 2

i) (10pts) Define a metric space. What does it mean for a function

$$f : (X, d) \to (X, d)$$

from a metric space to itself to be an isometry?

ii) (10 pts) Give the standard Euclidean metric on $\mathbb{R}^2$, and verify that

$$(x, y) \mapsto \left(\frac{x + \sqrt{3}y}{2}, \frac{-\sqrt{3}x + y}{2} \right)$$

preserved the metric.

iii) (13 pts) Define reflection in a line in the Euclidean plane, and show that reflection is an isometry. You may freely use the postulates given on page 6.

Question 3

Fix the lattice $\Lambda = \mathbb{Z}^2 \subset \mathbb{R}^2$.

i) (10 pts) Define a lattice polygon and an elementary triangle for the lattice λ

ii) (10 pts) Describe how a triangulation of a lattice polygon can be refined to an elementary triangulation. Be sure to justify that the process terminates.

iii) (13 pts) State Pick's Theorem, being sure to explain all notation. On graph paper, draw the triangle with vertices $(0, 0)$, $(2, 3)$, and $(7, -2)$, and use Pick's theorem to compute the area.

In an actual exam, a diagram would likely be provided

Question 4

i) (10 pts) Let AB be a hyperbolic line segment. Show that there exists a point C such that ABC is a equilateral hyperbolic triangle.

ii) (8 pts) Define the hyperbolic angle between two intersecting hyperbolic lines

iii) (10 pts) State Lambert's theorem for the area of a hyperbolic triangle and hence conclude the angles in a hyperbolic equilateral triangle are less than $\frac{\pi}{3}$.

iv) (7 pts) Define an ideal triangle and explain why it has 0 internal angles.

Propositions from Euclid's Elements

1. One can construct an equilateral triangle on any base.
2. One can construct a line of length equal to a given line, from a given point
3. One can construct a line of length equal to the difference of two given line segments
4. If two triangles have two sides equal to two sides, and the angles contained are equal, then the triangles are equal (SAS)
5. In an isosceles triangle, the angles at the base are equal

8. If two triangles have three sides equal to three sides, the triangles are equal (SSS)

16. In angle triangle, if one of the sides is extended, the exterior angle is greater than either of the interior opposite angles
17. In any triangle, the sum of any two angles is less than two right angles.

23. One can construct an angle equal to a given angle on a given straight line and at a point on it.

26. If two triangles have two angles equal to two angles, and the sides inbetween are equal, then the triangles are equal (ASA)
27. If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another

29. A straight line falling on two parallel straight lines makes the alternate angles equal to one another and the exterior angle equal to the interior opposite angle

31. Given a straight line and a point not on the line, one can draw a parallel line through the point