Euclidean and non-Euclidean Geometry - Sample Exam 1 - 2026

Section A

Q1: i)

- To draw a straight line from any point to any point
- To produce a finite straight line continuously in a straight line
- To describe a circle with any centre and distance
- That all right angles are equal
- That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines - if produced indefinitely - meet on that side.

Playfair's reformulation is also acceptable

ii) Every pair of antipodal points on S^1 uniquely determines a line and hence a point in $\mathbb{P}^1$. We define an equivalence relation on S^1 by $x \sim y$ if x and y are antipodal. Thus, we have a bijection $S^1/\sim \to \mathbb{P}^1$ sending the equivalence class of a point to the line through it and the origin. Choosing a representative of each class with positive y-coordinate and identifying the points $(\pm 1, 0)$ as shown, we get a bijection

$$S^1 \to S^1/\sim \to \mathbb{P}^1$$

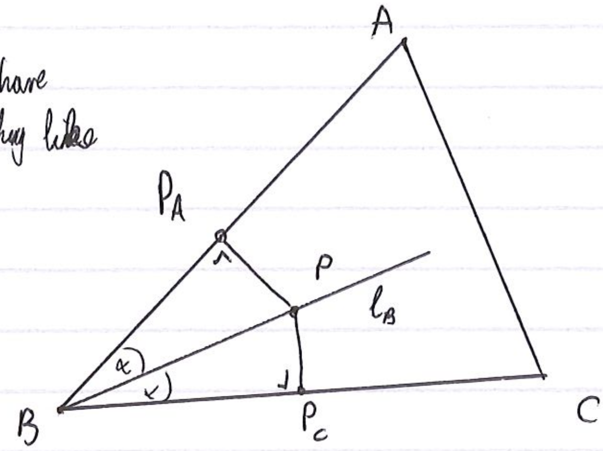
iii) If T is a triangle with internal angles α, β, γ, then

$$\text{Area}(T) = r^2 (\alpha + \beta + \gamma - \pi)$$

or $\quad \alpha + \beta + \gamma = \pi + \frac{1}{r^2} \text{Area}(T)$

iv) Let C be a circle with centre O and radius r. Then inversion is a map $\iota_C : \mathbb{R}^2 \setminus O \to \mathbb{R}^2 \setminus O$ defined by $\iota_C(P)$ being the unique point P' on the line from O through P such that $|OP||OP'| = r^2$

Q2 i) We have something like



· ~~Using Proposition~~ As $\angle P_ABP = \angle ABP = \angle CBP = \angle P_CBP$

and $\angle BP_AP = \angle BP_CP$, we have

$$\angle BPP_A = \pi - \angle P_ABP - \angle BP_AP$$

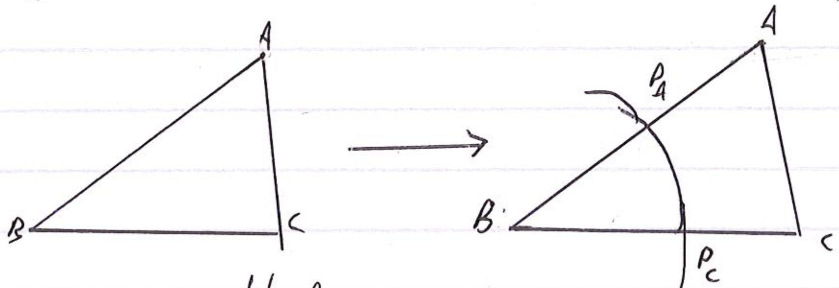
$$= \pi - \angle P_cBP - \angle BP_cP = \angle BPP_C$$

· Then as $|PB| = |PB|$, we have ASA congruence, and hence

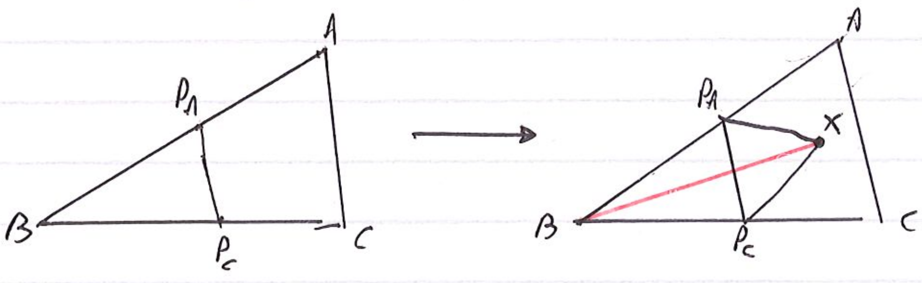
$$\triangle BPP_A = \triangle BPP_C$$

$$\Rightarrow |PP_A| = |PP_c| \text{ and } |BP_A| = |BP_c|$$

ii) · Construct a circle of any radius centred at B and let P_A and P_c be the intersections with AB and CB respectively



· Construct an ~~equivalent~~ equilateral triangle with base P_AP_c and apex X



We claim BX is the angle bisector. It suffices to show

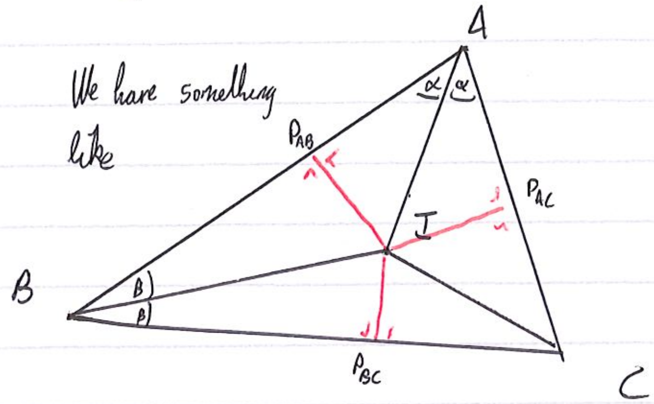
$$\angle P_ABX = \angle P_cBX$$

as the rest follows from (i). We have $|P_AB| = |P_cB|$, $|P_AX| = |P_cX|$ and $|XB| = |XB|$ by construction. Then $\triangle P_ABX = \triangle P_cBX$ and the claim follows

iii) $l_A = \{P \mid \angle PAB = \angle PAC\} \cup \{A\}$

$l_c = \{P \mid \angle PCA = \angle PCB\} \cup \{C\}$ Not needed for full marks.

We have something like



Drawing perpendiculars from I to each side as shown, we have congruences, as in i)

$$\Delta BP_{BC}I = \Delta BP_{AB}I$$
$$\Delta AP_{AC}I = \Delta AP_{AB}I$$

$\Rightarrow |IP_{BC}| = |IP_{AB}| = |IP_{AC}|$

By Pythagoras' Theorem, we have

$$|P_{BC}C|^2 = |IC|^2 - |P_{BC}I|^2$$
$$= |IC|^2 - |P_{AC}I|^2 = |P_{AC}C|^2$$

$\Rightarrow |P_{BC}C| = |P_{AC}C|$

So $|IP_{BC}| = |IP_{AC}|$, $|CP_{BC}| = |CP_{AC}|$, $|IC| = |IC|$

$\Rightarrow \Delta ICP_{BC} = \Delta ICP_{AC}$

$\Rightarrow \angle ICP_{BC} = \angle ICP_{AC}$

$\Rightarrow I$ is on l_c.

(iv) The circle centred at I of radius $|IP_{AC}|$ is the desired circle, as the radius $IP_{\bullet\bullet}$ is perpendicular to the corresponding side.

Q3. i) The image $\rho_S(P)$ of a point P is the intersection of the line NP with the plane $z=-1$, where $N=(0,0,1)$

Such a line is given by the set

$$l = \{ (1-t)(0,0,1) + t(x_0,y_0,z_0) \mid t \in \mathbb{R} \}$$

if $P = (x_0, y_0, z_0)$

$$\Rightarrow l = \{ (tx_0, ty_0, 1-t+tz_0) \mid t \in \mathbb{R} \}$$

This intersects $z=-1$ at $1-t+tz_0 = -1$

$$\Leftrightarrow t = \frac{2}{1-z_0}$$

Thus

$$\rho_S(x_0,y_0,z_0) = \left(\frac{2x_0}{1-z_0}, \frac{2y_0}{1-z_0}, -1 \right)$$

ii) Let C be a great circle and let ϖ be the corresponding plane. If $N \in \varpi$, then $\rho_S(C)$ is just the intersection of $\{z=-1\}$ and ϖ which is a Euclidean line.

If $N \notin \varpi$, suppose $(u,v,-1) \in \rho_S(C)$. Then

$$\rho_S^{-1}(u,v,-1) \in \varpi$$

$$\Rightarrow \left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{u^2+v^2-4}{u^2+v^2+4} \right) \in \varpi.$$

If $\varpi = \{ (x,y,z) \mid Ax+By+Cz=0 \}$, then we have

$$\frac{4Au + 4Bv + C(u^2+v^2-4)}{u^2+v^2+4} = 0$$

$$\Rightarrow C(u^2+v^2-4) + 4Au + 4Bv = 0$$

As $N \notin \varpi$, $C \neq 0$, so we get

$$u^2 + \frac{4A}{C}u + v^2 + \frac{4B}{C}v - 4 = 0$$

$$\Rightarrow \left(u + \frac{2A}{C}\right)^2 + \left(v + \frac{2B}{C}\right)^2 = 4 + \frac{4A^2+4B^2}{C^2}$$

which is a Euclidean circle

ii) Clearly C must lie on $\{x=0\}$, as reflection in $\{x=0\}$ swaps A and B. ~~and so must send~~ C

The hyperbolic circle with centre B and radius $d_{\mathbb{H}}(A,B)=R$ is the Euclidean circle with centre $(x, y\cosh R)$ and radius $(y\sinh R)$

This has equation $(u-x)^2+(v-y\cosh R)^2 = y^2\sinh^2 R$
which intersects the y-axis at $u=0$, and v satisfying

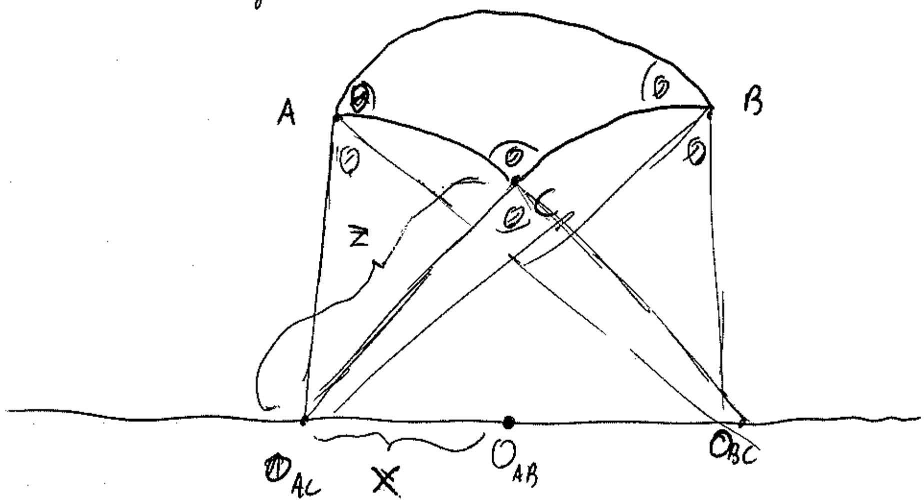
$$x^2+v^2-2vy\cosh R+y^2\cosh^2 R - y^2\sinh^2 R = 0$$

$$\iff v^2-2vy\cosh R + x^2+y^2 = 0$$

$$\iff v = y\cosh R \pm \sqrt{y^2\cosh^2 R - x^2 - y^2}$$

$$\iff v = y\cosh R \pm \sqrt{y^2\sinh^2 R - x^2}$$

Lets take $C = (0, y\cosh R - \sqrt{y^2\sinh^2 - x^2})$
Then we have the picture, where $O_{X,Y}$ is the centre of the semicircle through XY



- As tangents are perpendicular to radius, $\overset{Hyp}{\angle} ACB = \overset{Euc}{\angle} O_{AC} C O_{BC}$
- As the triangle is equilateral, $\overset{Hyp}{\angle} ACB = \overset{Hyp}{\angle} CAB = \overset{Hyp}{\angle} BAC = \theta$,

so it is enough to find $\overset{Euc}{\angle} O_{AC} C O_{BC} = \theta$

- By cosine rule $\cos\theta = \dfrac{|O_{AC}O_{BC}|^2 - 2|O_{AC}C|^2}{2|O_{AC}C|^2} = \dfrac{(2X)^2}{2Z^2} - 1$

$O_{AC} = (X, 0)$ s.t $(-x-X)^2+(y-0)^2 = Z^2$
and $(0-X)^2 + (y\cosh R - \sqrt{y^2\sinh^2 R - x^2} - 0)^2 = Z^2$

iii) Let C be our circle, and ϖ the corresponding plane. The case where $N \in \varpi$ is identical, so suppose $N \notin \varpi$. Then

$$\varpi = \{(x,y,z) \mid Ax + By + Cz = D\}$$

with $C \neq D$. As before, we have that if $(u,v,-1) \in \rho_s(C)$, then

$$\frac{4Au + 4Bv + C(u^2+v^2-4)}{u^2+v^2+4} = D$$

$$\Leftrightarrow \quad (C-D)(u^2+v^2) + 4Au + 4Bv = 4(C+D)$$

$$\Leftrightarrow \quad u^2 + \frac{4Au}{C-D} + \frac{4Bv}{C-D} + v^2 = \frac{4C+D}{C-D}$$

$$\Leftrightarrow \quad \left(u + \frac{2A}{C-D}\right)^2 + \left(v + \frac{2B}{C-D}\right)^2 = 4\left(\frac{A^2+B^2+C^2-D^2}{(C-D)^2}\right)$$

which is either empty or a Euclidean circle. It's not empty, as stereographic projection is a bijection, so it is a circle.

iv) If $\rho_s(C)$ is a Euclidean line, C is a spherical circle through N. Such a circle is uniquelly determined by a plane through N, except the plane $z = +1$. Every such plane is determined by a normal vector. $\{z = +1\}$ has normal vectors along the z-axis, so excluding this we get the deserded surjection

$$\vec{v} \mapsto \text{Normal Plane Through } N \Leftrightarrow \text{Circles through } N$$
$$\Leftrightarrow \text{Given set.}$$

Q4 i) Given points A, B, we construct (hyperbolic) circles C_A, C_B with (hyperbolic) centres A and B and (hyperbolic) radius $d_H(A,B)$. Exactly as in the Euclidean case, these will intersect in a point C s.t

$$d_H(A,B) = d_H(A,C) = d_H(B,C)$$

It will not be a Euclidean equilateral triangle, as the Euclidean circles centred at A and B will not align with the hyperbolic circles

With a bit of algebra we find

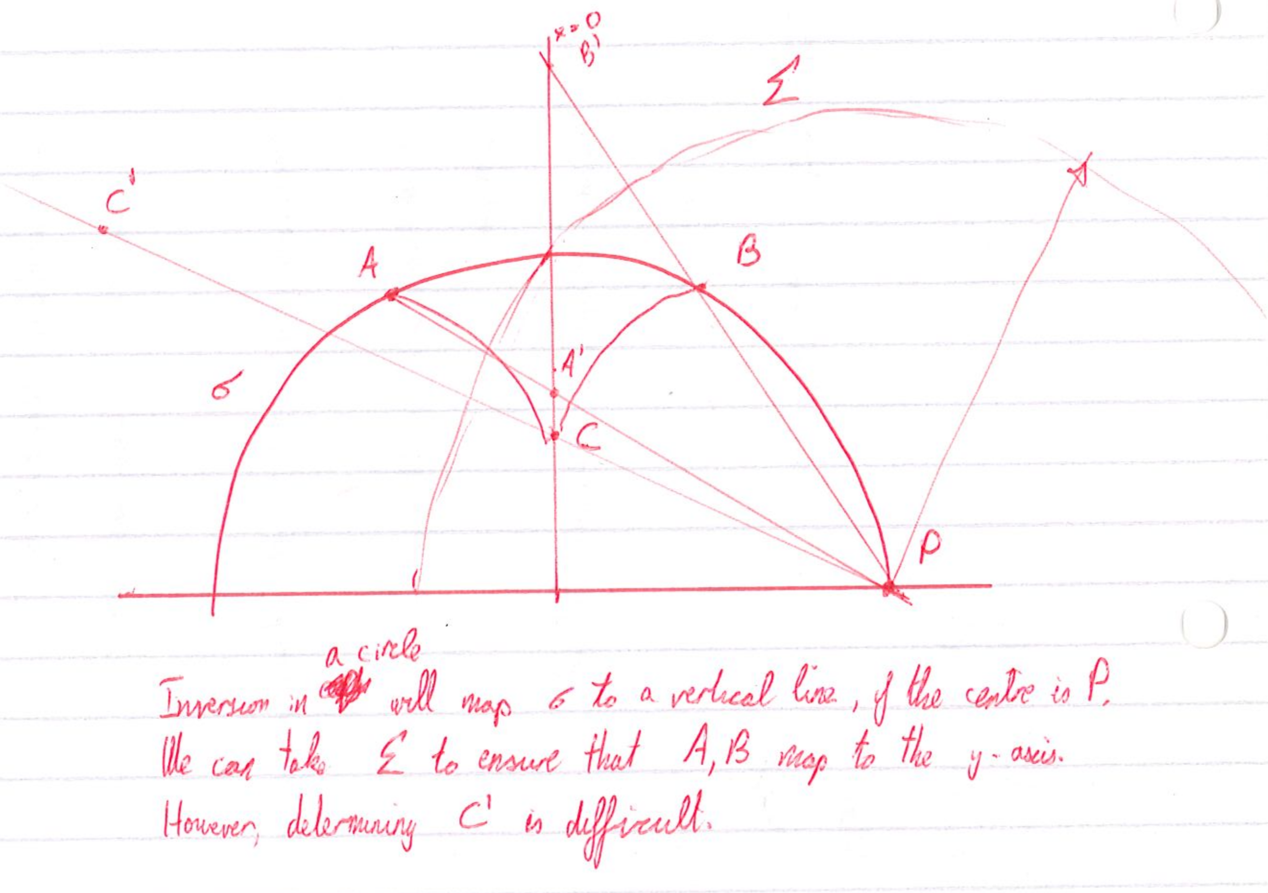
$$X = \frac{y_0^2\sinh^2 R - x_0^2 - 2y_0\cosh R\sqrt{y_0^2\sinh^2 R - x_0^2}}{x_0}$$

$$Z^2 = x_0^2 + 2x_0 X + X^2 + y_0^2$$

$$\Rightarrow \quad \cos\Theta = \frac{4X^2}{2Z^2} - 1 = 2\frac{X^2}{x_0^2 + 2x_0X + X^2 + y_0^2} - 1$$

$$= \frac{X^2 - 2x_0X - x_0^2 - y_0^2}{X^2 + 2x_0X + x_0^2 + y_0^2}$$

Or, use an inversion



Inversion in a circle will map σ to a vertical line, if the centre is P. We can take Σ to ensure that A, B map to the y-axis. However, determining C' is difficult.

(iii) $$\text{Area}(ABC) = \pi - 3\Theta$$
$$= \pi - 3\cos^{-1}\left(\frac{X^2 - 2x_0X - x_0^2 - y_0^2}{X^2 + 2x_0X + x_0^2 + y_0^2}\right)$$

Note: This question would be simplified in an exam setting

Section B

Q1 i) As for 1Ai

ii) An inner product on a vector space V is a bilinear function $I : V \times V \to \mathbb{R}$

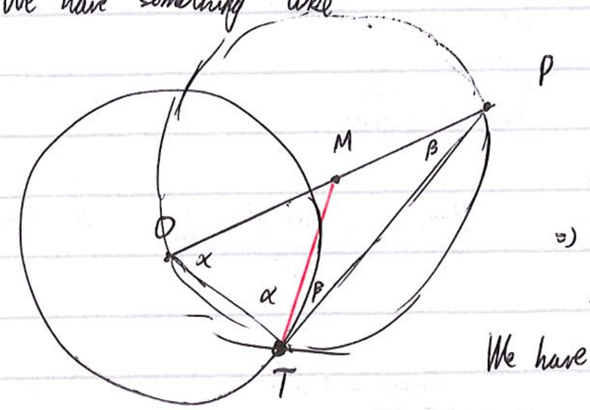
s.t $\quad I(x,y) = I(y,x)$

$I(x,x) \geq 0$ with $I(x,x) = 0$ iff $x = 0$.

iii) $\text{Area}(T) = r^2(\pi + \alpha + \beta + \gamma)$ where α, β, γ are the internal angles of T.

iv) As for 1Aiv

Q2: i) We have something like



As $|OM| = |TM| = |PM|$

ΔOMT and ΔTMP are isoceles

$\Rightarrow \angle OTM = \angle TOM =: \alpha$

$\angle TPM = \angle TMP =: \beta$

We have

$\alpha + \alpha + \beta + \beta = \pi$

$\Rightarrow 2\alpha + 2\beta = \pi \Rightarrow \alpha + \beta = \frac{\pi}{2}$

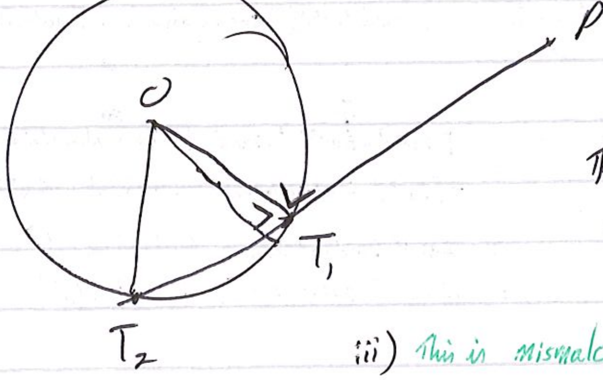
$\Rightarrow \angle OTP = \frac{\pi}{2}$

ii) Suppose it intersects the circle twice. By the same argument

$\angle OT_1T_2 = \angle OT_2T_1 = \frac{\pi}{2}$

$\Rightarrow$ The triangle OT_1T_2 has internal angle greater than $\frac{\pi}{2} + \frac{\pi}{2} = \pi$, impossible

Hence, the line intersects the circle once.



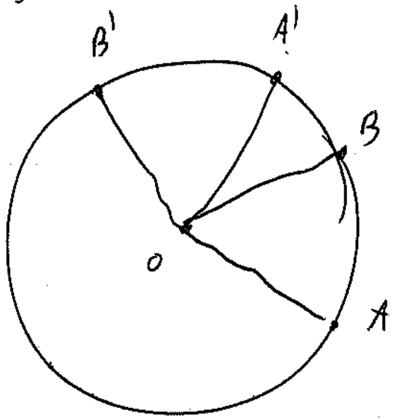
iii) This is mismatched the the question

Draw a circle centred at A and at B of radius $|AB|$. Join the intersections. This line intersects AB at the midpoint

Q3: i) An isometry of a metric space (X,d) is a bijection $f: X \to X$ s.t $d(f(x), f(y)) = d(x,y)$ for all $x,y \in X$.

We have $d(f^{-1}(x), f^{-1}(y)) = d(f(f^{-1}(x)), f(f^{-1}(y)))$ as f is an isometry. As $f(f^{-1}(x)) = x$, we get
$$d(f^{-1}(x), f^{-1}(y)) = d(x,y)$$
$\Rightarrow f^{-1}$ is an isometry

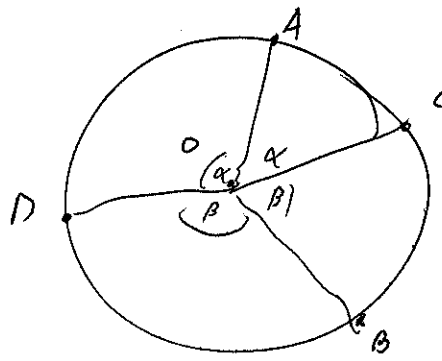
ii) This can be shown by proving $\Delta AOB = \Delta A'OB'$ or using rotation matrices, or



We have $\angle AOA' = \angle BOB' = \Theta$
$$\Rightarrow \angle A'OB' = \angle BOB' - \angle A'OB$$
$$= \Theta - \angle A'OB$$
$$= \Theta - (\Theta - \angle AOB)$$
$$= \angle AOB$$
$\Rightarrow$ Isometry

iii) ~~Given two points that are~~
We claim that, given A, B not diametrically opposite, there is at most one C s.t $d(A,C) = \alpha$, $d(B,C) = \beta$ for given α, β.

The condition $d(A,C) = \alpha$ determines 2 points one on either side of A. Either $d(B,C) = \beta$ uniquely determines one of these points, no such point exists, or $2\alpha + 2\beta = 2\pi$
$\Rightarrow \alpha + \beta = \pi$
$\Rightarrow A, B$ are opposite



iv) Given an isometry f we can find a rotation R s.t $R(A) = f(A)$ for some A. Picking B not opposite, then either $R(B) = f(B)$ and we are done, or $R(B)$ is the reflection of $f(B)$ in the diameter through A

Q4: i) $\mathbb{RP}^2 = \{ \cancel{B} \subseteq \mathbb{R}^3 \mid \cancel{B}$ is a ~~plane~~ line through the origin $\}$

ii) We rescale them also so that the z-coordinate is 1

$(1,2,3) \sim (\frac{1}{3}, \frac{2}{3}, 1)$

$(2,2,1) \sim (2,2,1)$ Only these two

$(5,10,15) \sim (\frac{1}{3}, \frac{2}{3}, 1)$

$(1,\frac{1}{2},\frac{3}{2}) \sim (\frac{2}{3}, \frac{1}{3}, 1)$

$(\frac{\pi}{2}, \pi, 3\pi) \sim (\frac{1}{6}, \frac{1}{3}, 1)$

iii) As the middle coordinate in non-zero, we will scale that to 1

$$\left.\begin{aligned} A &= (1,7,0) \sim (\tfrac{1}{7}, 1, 0) \\ B &= (\cancel{0},1,1) \sim (0,1,1) \\ C &= (2,1,-13) \sim (2,1,-13) \\ D &= (1,-7,-14) \sim (\tfrac{-1}{7}, 1, -2) \end{aligned}\right\} \text{These lie on the line } 7x + z - 1 = 0$$

The cross ratio is $\frac{|C-A|}{|C-B|} \cdot \frac{|D-B|}{|D-C|}$

where $|C-A| = \sqrt{(2-\frac{1}{7})^2 + (-13)^2} = \sqrt{\frac{169}{49} + 169} = \frac{13}{7}\sqrt{50}$

$|C-B| = \sqrt{2^2 + (-13-1)^2} = \sqrt{200} = 2\sqrt{50}$

$|D-B| = \sqrt{(\frac{-1}{7})^2 + (-2-1)^2} = \sqrt{\frac{1}{49} + 9} = \frac{1}{7}\sqrt{442}$

$|D-C| = \sqrt{(\frac{-1}{7}-2)^2 + (-2+13)^2} = \sqrt{\frac{225}{49} + 121} = \frac{1}{7}\sqrt{6154}$

$\Rightarrow$ Cross-ratio $= \frac{13}{14} \cdot \frac{\sqrt{442}}{\sqrt{6154}} = \frac{13}{14} \frac{\sqrt{2 \cdot 13 \cdot 17}}{\sqrt{2 \cdot 17 \cdot 181}} = \frac{13}{14}\sqrt{\frac{13}{181}}$