



Coláiste na Tríonóide, Baile Átha Cliath
Trinity College Dublin
Ollscoil Átha Cliath | The University of Dublin

Faculty of Science, Technology, Engineering and Mathematics
School of Mathematics

Sample Exam 1

Hilary Term

Euclidean and non-Euclidean Geometry

Day

Place

Time

Dr. Adam Keilthy

Instructions to candidates:

Attempt any three questions from the appropriate section. Undergraduates should attempt only Section A. Students taking the PME should attempt only Section B.

If you attempt all four questions, only your best three will be considered in your grade. Each question is worth 33 points. One point will be awarded for general presentation.

Unless stated otherwise, you may use all statements given in lectures without proof, but must clearly justify that the assumptions of the statement are fulfilled.

Additional instructions for this examination:

Formula and tables are available from the invigilators if required. A list of Euclidean Propositions is provided on the final page

You may use a non-programmable calculator. Please indicate the make and model of your calculator on each answer book used.

You may not start this examination until you are instructed to do so by the Invigilator.

Section A - Undergraduate

Question 1A

- i) (10 pts) State the 5 postulates of Euclidean geometry.
- ii) (8 pts) Give a bijection between the projective line \mathbb{P}^1 and the circle S^1 .
- iii) (8 pts) State Girard's Theorem for the area of a spherical triangle.
- iv) (7 pts) Give a geometric definition of inversion in a circle.

Question 2A

Using only the propositions provided, along with the postulates and common notions of Euclidean geometry, show the following.

Let ABC be a non-degenerate Euclidean triangle, and define the angle bisector to be the set

$$\ell_B = \{P \mid \angle PBA = \angle PBC\} \cup \{B\}$$

- i) (10 pts) Let $P \in \ell_B$, $P \neq B$. Draw a perpendicular from P to AB , intersecting AB at P_A , and similarly construct P_C . Show that

$$|PP_A| = |PP_C|, \quad \text{and} \quad |P_AB| = |P_CB|.$$

- ii) (8pts) Hence give a construction of ℓ_B using only a straight-edge and compass. Be sure to proof that it works.
- iii) (10 pts) Define the angle bisectors ℓ_A and ℓ_C . Let I be the intersection of ℓ_A and ℓ_B . Show that $I \in \ell_C$.
- iv) (5 pts) Hence conclude that there is a circle C tangent to each side of ABC .

Question 3A

- i) (10pts) Define the stereographic projection ρ_S of the unit sphere onto the plane $z = -1$, and derive in terms of coordinates.
- ii) (10 pts) Let C be a great circle. Show that $\rho_S(C)$ is either a Euclidean circle or a Euclidean line.
- iii) (10 pts) Let C be an arbitrary spherical circle. Show that $\rho_S(C)$ is either a Euclidean circle or a Euclidean line.
- iv) (3 pts) Let $\mathbb{R}\vec{k} \subset \mathbb{R}^3$ denote the z -axis. Give a surjective map from $\mathbb{R}^3 \setminus \mathbb{R}\vec{k}$ to the set

$$\{C \subset \mathbb{S}_1^2 \mid \rho_S(C) \text{ is a Euclidean line}\}$$

and describe the inverse image of an arbitrary point.

Hint: It may be helpful to start by identifying C with an affine subspace of \mathbb{R}^3

Question 4A

- i) (10 pts) Show that, given a hyperbolic line AB , there exists a point C such that ABC is both an equilateral hyperbolic triangle and an equilateral Euclidean triangle.
- ii) (15 pts) Suppose that

$$A = (-x, y), \quad B = (x, y)$$

and determine the coordinates of C . Hence compute the hyperbolic angles

$$\angle ABC, \angle ACB, \angle BAC.$$

- iii) (8 pts) Hence determine the hyperbolic area of the triangle ABC as a function of (x, y) .
If very bored, compute the area using

$$\text{Area}(ABC) = \iint_{ABC} \frac{dx dy}{y(t)^2}$$

Section B - Postgraduate

Question 1

- i) (10 pts) State the 5 postulates of Euclidean geometry.
- ii) (8 pts) Define an inner product on a vector space.
- iii) (8 pts) State Girard's Theorem for the area of a spherical triangle.
- iv) (7 pts) Give a geometric definition of inversion in a circle.

Question 2B

Using only the propositions provided, along with the postulates and common notions of Euclidean geometry, show

- i) (15 pts) Let C be a circle with centre O , and P a point outside C . Let M be the midpoint of OP and let T be an intersection of the circle S with centre M and radius MP with C . Show that $\angle OTP = \frac{\pi}{2}$.
- ii) (10 pts) Hence conclude that the line PT , even if extended indefinitely, intersects the circle exactly once.
- iii) (8 pts) Using part (ii), give a straight-edge and compass construction of the perpendicular bisector of a line segment.

Question 3B

- i) (10 pts) Define an isometry of a metric space, and show that the inverse of an isometry is an isometry.
- ii) (10 pts) Let \mathcal{C} be a circle with radius 1 and centre O . We define a metric on \mathcal{C} by

$$d(A, B) = \angle AOB.$$

The rotation around O by θ of a point A is the unique point $A' \in \mathcal{C}$ such that

$$\angle AOA' = \theta$$

(measured counterclockwise from A).

Show that rotation around O by θ is an isometry for this metric. You may use either analytic or synthetic methods.

- iii) (10 pts) Show that every isometry of \mathcal{C} is uniquely determined by the image of two points A and B that are not diametrically opposite.
- iv) (3 pts) Hence conclude that every isometry is a combination of rotation around O and reflection in a diameter

Question 4B

- i) (10 pts) Define the real projective plane \mathbb{RP}^2 .
- ii) (10 pts) Which of the following define the same points in \mathbb{RP}^2 ?

$$(1, 2, 3), (2, 2, 1), (5, 10, 15), \left(1, \frac{1}{2}, \frac{3}{2}\right), \left(\frac{\pi}{2}, \pi, 3\pi\right),$$

- iii) (13 pts) By picking appropriate homogeneous coordinates, verify that these three points are collinear and compute their cross ratio.

$$(1, 7, 0), (0, 1, 1), (2, 1, -13), (1, -7, -14)$$

Propositions from Euclid's Elements

1. One can construct an equilateral triangle on any base.
2. One can construct a line of length equal to a given line, from a given point
3. One can construct a line of length equal to the difference of two given line segments
4. If two triangles have two sides equal to two sides, and the angles contained are equal, then the triangles are equal (SAS)
5. In an isosceles triangle, the angles at the base are equal
8. If two triangles have three sides equal to three sides, the triangles are equal (SSS)
23. One can construct an angle equal to a given angle on a given straight line and at a point on it.
26. If two triangles have two angles equal to two angles, and the sides inbetween are equal, then the triangles are equal (ASA)
27. If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another
29. A straight line falling on two parallel straight lines makes the alternate angles equal to one another and the exterior angle equal to the interior opposite angle
31. Given a straight line and a point not on the line, one can draw a parallel line through the point

Book 3

18. The tangent line to a circle is perpendicular to its radius
19. A line perpendicular through a circle perpendicular to the radius at a point of intersection is a tangent line.