MAU22302/33302 - Euclidean and non-Euclidean Geometry

Tutorial Sheet 9

Trinity College Dublin

Course homepage

Exercise 1 *Cross ratio and infinity*

i) Compute the following cross ratios

$$[1, 2, 3, 4],\ [2, 5, 8, 20],\ [4, \infty, 0, 6]$$

ii) Compute $[x_2, x_1, x_4, x_3]$ and $[x_3, x_4, x_1, x_2]$ in terms of $[x_1, x_2, x_3, x_4]$.

iii)

iv) Determine expressions for

$$[\infty, x_2, x_3, x_4],\ [x_1, \infty, x_3, x_4],\ [x_1, x_2; \infty, x_4],\ [x_1, x_2, x_3, \infty]$$

in terms of the finite coordinates

v) Determine a formula for the cross ratio in terms of homogeneous coordinates

$$\big[[x_1 : y_1], [x_2 : y_2], [x_3 : y_3], [x_4 : y_4]\big] =?$$

Start with the case of finite coordinates, and check it works for one with $y_i = 0$ for some i.

Solution 1

i) Using

$$[x_1, x_2, x_3, x_4] = \frac{(x_3 - x_1)(x_4 - x_2)}{(x_3 - x_2)(x_4 - x_1)}$$

we find

$$\begin{aligned}
[1, 2, 3, 4] &= \frac{(2)(2)}{(1)(3)} = \frac{4}{3}, \\
[2, 5, 8, 20] &= \frac{(6)(15)}{(3)(18)} = 5 \\
[4, \infty, 0, 6] &= \lim_{x\to\infty} \frac{(-4)(6-x)}{(-x)(2)} \\
&= \lim_{x\to\infty} 2\frac{6-x}{x} = 2\lim_{x\to\infty} \frac{\frac{6}{x} - 1}{1} = -2
\end{aligned}$$

ii) As any permutation can be obtained by repeated swapping of adjacent pairs, we will just consider three cases

$$\begin{aligned}
[x_2, x_1, x_4, x_x] &= \frac{(x_4 - x_2)(x_3 - x_1)}{(x_4 - x_1)(x_3 - x_2)} = [x_1, x_2, x_3, x_4] \\
[x_3, x_4, x_1, x_2] &= \frac{(x_1 - x_3)(x_2 - x_4)}{(x_1 - x_4)(x_2 - x_3)} = [x_1, x_2, x_3, x_4].
\end{aligned}$$

Labelling the vertices of a square by $1, 2, 4, 3$ in that order, we see that these reorderings correspond to (even) symmetries of the square! In fact, these two symmetries and their composition are precisely the permutations preserving the cross ratio.

iii) We compute

$$[\infty, x_2, x_3, x_4] = \lim_{x\to\infty} \frac{\left(\frac{x_3}{x} - 1\right)(x_4 - x_2)}{(x_3 - x_2)\left(\frac{x_4}{x} - 1\right)} = \frac{x_4 - x_2}{x_3 - x_2}$$

Using the symmetries of the previous part (or by direct computation) we find

$$
\begin{aligned}
[x_1, \infty, x_3, x_4] = [\infty, x_1, x_4, x_3] &= \frac{x_3 - x_1}{x_4 - x_1} \\
[x_1, x_2, \infty, x_4] = [\infty, x_4, x_1, x_2] &= \frac{x_2 - x_4}{x_1 - x_4} \\
[x_1, x_2, x_3, \infty] = [x_2, x_1, \infty, x_3] &= \frac{x_1 - x_3}{x_2 - x_3}.
\end{aligned}
$$

and so we can just "omit" the terms with ∞ appearing.

iv) If all the points are finite, then we can rescale so that we have coordinates $z_i = \frac{x_i}{y_i}$, in which case the cross ratio is

$$
\frac{\left(\frac{x_3}{y_3} - \frac{x_1}{y_1}\right)\left(\frac{x_4}{y_4} - \frac{x_2}{y_2}\right)}{\left(\frac{x_3}{y_3} - \frac{x_2}{y_2}\right)\left(\frac{x_4}{y_4} - \frac{x_1}{y_1}\right)}
$$

which simplifies to

$$
\frac{(x_3y_1 - x_1y_3)(x_4y_2 - x_2y_4)}{(x_3y_2 - x_2y_3)(x_4y_1 - x_1y_4)}
$$

If one of the points is infinite, say $y_1 = 0$, then the formula gives

$$
\frac{(-x_1y_3)(x_4y_2 - x_2y_4)}{(x_3y_2 - x_2y_3)(-x_1y_4)} = \frac{y_3(x_4y_2 - x_2y_4)}{y_4(x_3y_2 - x_2y_3)}
$$

while the cross ratio from the previous part gives

$$
\frac{\left(\frac{x_4}{y_4} - \frac{x_2}{y_2}\right)}{\left(\frac{x_3}{y_3} - \frac{x_2}{y_2}\right)}
$$

which simplifies to exactly the given formula. Symmetries of the cross ratio then imply all other cases.

Exercise 2 *Constructing projective transformations*

Using the cross ratio, determine matrix associated to the following projective transformations $\mathbb{RP}^1 \to \mathbb{RP}^1$, given by specifying the images of three points:

i)
$$[1:1] \mapsto [2:1],\ [3:1] \mapsto [5:1],\ [5:1] \mapsto [8:1]$$

ii)
$$[4:2] \mapsto [6:3],\ [15:3] \mapsto [8:2],\ [100:25] \mapsto [255:15]$$

iii)
$$[1:0] \mapsto [0:1],\ [3:6] \mapsto [1:1],\ [8:5] \mapsto [3:1]$$

Solution 2

i) We know any projective transformation preserves the cross ratio, and so for a finite point $[x:1]$, we have that

$$2\frac{x-3}{x-1} = \frac{(5-1)(x-3)}{(5-3)(x-1)} = \frac{(8-2)(\rho(x)-5)}{(8-5)(\rho(x)-2)} = 2\frac{\rho(x)-5}{\rho(x)-2}$$

Solving for $\rho(x)$, we find

$$\rho(x) = \frac{3}{2}x + \frac{1}{2}$$

and so

$$[\rho(x):1] = [\frac{3}{2}x + \frac{1}{2} : 1] = [3x+1:2] = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} [x:1].$$

For the infinite point, we must have

$$[2 = [1,3,5,\infty] = 2\frac{\rho(\infty)-5}{\rho(\infty)-2}$$

which has no solution unless $\rho(\infty) = \infty$. We can verify that

$$\begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} [1:0] = [3:0] = [1:0]$$

and so this matrix does indeed realised the given projective transformation.

ii) We actually don't need to check the infinite point as a separate case. We know that a projective transformation is uniquely determined by the image of three points, so as long as we find a single matrix (up to rescaling), that must be the matrix of the projective transformation. With that in mind, we rescale all our points to be in the form $[\alpha : 1]$ and then will compare cross ratios. We have

$$[2:1] \mapsto [2:1],\ [5:1] \mapsto [4:1],\ [4:1] \mapsto [17:1]$$

and so

$$\frac{(4-2)(x-5)}{(4-5)(x-2)} = \frac{(17-2)(\rho(x)-4)}{(17-4)(\rho(x)-2)}.$$

Solving for $\rho(x)$, we find

$$\rho(x) = \frac{116x - 380}{41x - 160}$$

and so

$$[\rho(x):1] = \left[\frac{116x-380}{41x-160} : 1\right] = [116x-380 : 41x-160] = \begin{pmatrix} 116 & -380 \\ 41 & -160 \end{pmatrix} [x:1]$$

gives us the desired matrix.

iii) Here we have a point being mapped to ∞, so we'll need to take a big of care. From an earlier part, we know how to compute the relevant cross ratios so the equality

$$[1, 1/2, 8/5, x] = [\infty, 1, 3, \rho(x)]$$

becomes

$$\frac{(8/5-1)(x-1/2)}{(8/5-1/2)(x-1)} = \frac{\rho(x)-1}{3-1}$$

which we can solve for $\rho(x)$ to find

$$[\rho(x):1] = \left[\frac{23x-17}{11x-11} : 1\right] = [23x-17 : 11x-11] = \begin{pmatrix} 23 & -17 \\ 11 & 11 \end{pmatrix} [x:1]$$

giving the desired matrix.

Exercise 3 *Bonus: the complex projective line*

Entirely analogously to the real projective line, we can define the complex projective line $\mathbb{CP}^1$ as the set of all (complex) lines in $\mathbb{C}^2$ through the origin, and identically define homogeneous coordinates, and points at infinity. We can also establish a bijection with a more familiar space. Determine this familiar space.

Hint: How many points at infinity are there? What does the rest of the space resemble? Where have we here the word projection before?

Solution 3

Exactly as for the real projective line, we can write

$$\mathbb{CP}^1 = \{[x : y] \mid x, y \in \mathbb{C},\ (x, y) \neq 0\}$$

where $[x : y] = [\lambda x : \lambda y]$ for all non-zero $\lambda \in \mathbb{C}$. In particular, we have

$$\begin{aligned}\mathbb{CP}^1 =& \{[x : 1] \mid x \in \mathbb{C}\} \sqcup \{[1 : 0]\}\\ =& \mathbb{C} \sqcup \infty.\end{aligned}$$

Thus we get a copy of the complex plane, with a singular point at infinity. Thinking back to spherical projections, we have a lovely bijection

$$\rho_S : \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$$

given by stereographic projection. Identifying $\mathbb{R}^2$ with $\mathbb{C}$, and extending the map to N, we obtain a bijection

$$\rho_S : \mathbb{S}^2 \to \mathbb{C} \sqcup \infty = \mathbb{CP}^1$$

giving a bijection with the sphere! This similarly lets up define topologies on the complex projective line, as points that are “close” to infinity in the projective line correspond exactly to points on the sphere that are close to the north pole. This identification (with topology) completes the complex plane to what is called the Riemann sphere, which plays a key role in complex analysis and the study of meromorphic functions.