MAU22302/33302 - Euclidean and non-Euclidean Geometry

Tutorial Sheet 8

Trinity College Dublin

Course homepage

Exercise 1 *Hyperbolic reflection*

We saw that in Euclidean geometry that the composition of two reflections was either a translation (if the lines did not intersect) or a rotation (if they did intersect). Does this hold hyperbolically?

1. Consider two intersecting semicircles C_1, C_2 centred on the x-axis. Is the composition $\iota_{C_2} \circ \iota_{C_1}$ a Euclidean rotation? If not, can we interpret it as a hyperbolic rotation?

   Hint: In the Euclidean case, we would be rotating points along a circle centred at the point of intersection

2. Consider two non-intersecting semicircles C_1, C_2 centred on the x-axis. Is the composition $\iota_{C_2} \circ \iota_{C_1}$ a Euclidean translation?

3. In complex coordinates, determine a formula for the composition $\iota_{C_2} \circ \iota_{C_1}$ if C_1 is centred at the origin and has radius r, and C_2 is centred at $(a, 0)$ and has radius R.

4. In the Euclidean case we would be translating along a line perpendicular to both lines of reflection. Does such a line even exist in the hyperbolic case? Can we interpret the composition as translation along this line?

Hint: Consider tangent lines from the centre of this hyperbolic line. Is there necessarily a point where the tangent lines to each circle are equal?

Solution 1

1. Clearly the point of intersection I of C_1 and C_2 is a fixed point, so we would expect this to be the centre of rotation. As inversion is an isometry, this means that points on a hyperbolic circle with centre I are mapped to points on the same circle (which also means every hyperbolic circle with centre I is perpendicular to the inversion circles!)

   Using SAS congruence, the isometry will also preserve angles between points on the circle, so it makes sense to interpret this as a rotation if it preserves the order of points on one of these circles.

   Pick a hyperbolic circle with centre I and any fixed radius. It will intersect C_1 twice. Tracking how these points move under the inversions, it is (mostly) clear to see that the cyclic of points is preserved, so this really is some sort of hyperbolic rotation. (Only mostly, as it is not always immediately obvious which circular arc is shorter)

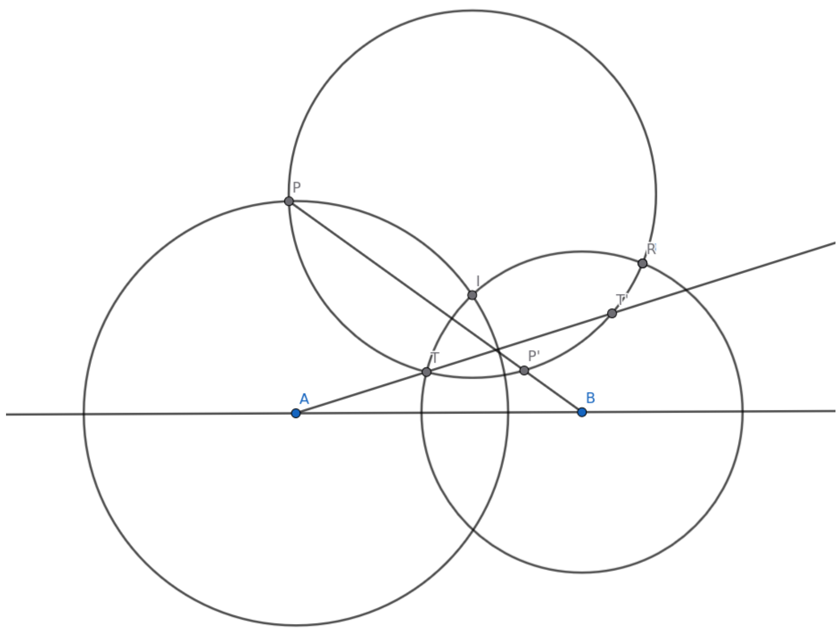
2. The answer is probably no, as Euclidean translation is not a hyperbolic isometry in general. But to see that it is definitely no, recall that inversion swaps the inside and outside of the circle. We take two points with y-coordinate larger than the radius of either circle and such that $|y_2 - y_1|$ is larger than both radii. These are outside both circles, so ι_{C_1} moves these points inside C_1 and ι_{C_2} then moves them inside C_2. As the images are both inside C_2, we cannot have performed a translation in the x-direction, and the difference in their y-coordinates is less than the radius of C_2, so this cannot have been a Euclidean translation.

3. Breaking the composition up into inversions in circles centred at 0 and translations, we find the composition is given by

   $$z \mapsto \frac{r^2}{\overline{z}} \mapsto \frac{r^2}{\overline{z}} - a \mapsto \frac{R^2}{\frac{r^2}{z} - a} \mapsto \frac{R^2}{\frac{r^2}{z} - a} + a$$

   which simplifies to

   $$\frac{R^2 z}{r^2 - az} + a$$



4. If there were a hyperbolic line ℓ perpendicular to both, then it would be fixed by both ι_{C_1} and ι_{C_2}, so we might have a case to make that we are translating along ℓ. Such a hyperbolic line always exists: given a point $(x, 0)$ strictly between C_1 and C_2, we can consider the lengths t_1, t_2 of the tangent lines from $(x, 0)$ to C_1 and C_2 respectively. These are continuous monotonic functions of x, with $t_1(x) > t_2(x)$ when $(x, 0)$ is on C_2, and $t_1(x) < t_2(x)$ when $(x, 0)$ is on C_1. Hence, there must be a (unique!) point when they are equal. This $(x, 0)$ is then the centre of a semicircle with radii tangent to C_1 and C_2. Hence this semicircle is perpendicular to C_1 and C_2.

   As for whether we can interpret the composition as translation along this line for points not on the line...debatable. What a hyperbolic translation is really depends on what is useful. The most common definition is an orientation preserving isometry with no fixed point, of which this is an example! But given that this perpendicular line is the unique line fixed by the composition, it is hard to really thing of this as translation.

Exercise 2 *Steps towards hyperbolic trigonometry*

Lets establish some simple trigonometry in $\mathbb{H}^2$

1. Suppose points $P, Q \in \mathbb{H}^2$ lie on a semicircle centred at the origin, with coordinates

$$P = (r\cos(\theta), r\sin(\theta)), \quad Q = (r\cos(\phi), r\sin(\phi)).$$

   Either by finding an appropriate isometry taking the hyperbolic line segment to a vertical line or by computing the length integral, show that

$$d_{\mathbb{H}}(P, Q) = \ln\left(\frac{\tan(\theta/2)}{\tan(\phi/2)}\right) = 2\tanh^{-1}\left(\frac{\sin((\theta-\phi)/2)}{\sin((\theta+\phi)/2)}\right)$$

   Hint: The semicircle containing P, Q is perpendicular to the x-axis. This is preserved under inversion. How can we get a line from inversion?

2. Show that

$$\cosh(d_{\mathbb{H}}(P, Q)) = 1 + \frac{\|P-Q\|^2}{2y_P y_Q}$$

   where $\|\cdot\|$ denotes the Euclidean distance, and y_X is the y-coordinate of the point X. Note that, as translation in the x-direction is a hyperbolic isometry, this gives a formula for all P, Q not on a vertical line (though it works for those too!).

   Hint: $\tan(\theta/2) = \frac{\sin(\theta)}{1+\cos(\theta)} = \frac{1-\cos(\theta)}{\sin(\theta)}$

3. Hence determine the hyperbolic distance between $(1, 1)$ and $(2, 7)$.

Remark 1. *Often, you see the hyperbolic distance expressed via the Euclidean norm*

$$d_{\mathbb{H}}(P, Q) = 2\tanh^{-1}\left(\frac{\|P-Q\|}{\|P-\tilde{Q}\|}\right)$$

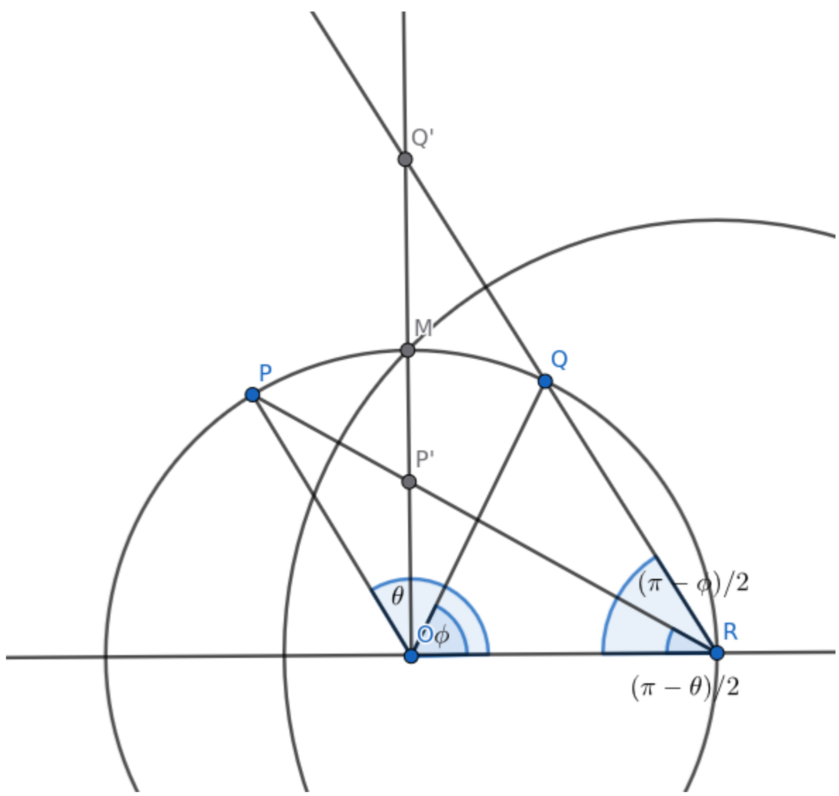
where $\tilde{Q}$ is the reflection of Q in the x-axis. Drawing a diagram, using the cosine rule and the double angle formula, it is not hard to see this is equivalent to the formula we arrived at.

Solution 2

1. In either method, we will assume $\theta \geq \phi$.

   To find an isometry, we note that as the semicircle σ containing P, Q is perpendicular to the x-axis, their images under inversion will be perpendicular. Choosing an inversion circle with centre on the x-axis means the x-axis will be fixed under inversion. Choosing an inversion circle whose centre is on σ ensures the semicircle will be mapped to a Euclidean line. Thus, choosing an intersection point R of σ with the x-axis as the centre of a circle C, we find that $\iota_C(PQ)$ will be a vertical line segment. By choosing C so that C contains the intersection of σ and the y-axis, we find $\iota_C(PQ)$ is a segment of the y-axis.

   We can therefore determine the images of P and Q as the intersections of the lines from P and Q to the centre of C with the y-axis, as shown.



   As the triangles POR and QOR are isosceles, we can easily compute their angles, and hence use Euclidean trigonometry to find

$$P' = (0, r\tan\left(\frac{\pi-\theta}{2}\right)) = (0, r\cot(\theta/2))$$
$$Q' = (0, r\tan\left(\frac{\pi-\phi}{2}\right)) = (0r\cot(\phi/2))$$

Thus, the hyperbolic distance is

$$\ln\left(\frac{r\cot(\phi/2)}{r\cot(\theta/2)}\right) = \ln\left(\frac{\tan(\theta/2)}{\tan(\phi/2)}\right).$$

If desired, this can be manipulated further using the product-to-sum formulae to get

$$\ln\left(\frac{\sin(\theta+\phi)+\sin(\theta-\phi)}{\sin(\theta+\phi)-\sin(\theta-\phi)}\right) = 2\tanh^{-1}\left(\frac{\sin(\theta-\phi)}{\sin(\theta+\phi)}\right)$$

It we wanted to compute this using the integral, we would parametrise the arc by

$$x(t) = r\cos(t), \quad y(t) = r\sin(t)$$

with $\phi \leq t \leq \theta$ (or the reverse). We find that the length is given by the integral

$$\int_\phi^\theta \frac{dt}{\sin(t)} = \ln\left(\tan\left(\frac{t}{2}\right)\right)\Big|_\phi^\theta$$

which we can compute by multiplying the integrand by $\frac{\sin(t)}{\sin(t)}$ and letting $u = \cos(t)$. Evaluating this, we find

$$\ln\left(\frac{\tan(\theta/2)}{\tan(\phi/2)}\right)$$

exactly as previously.

2. We have that

$$d_{\mathbb{H}}(P,Q) = \ln\left(\frac{\tan(\theta/2)}{\tan(\phi/2)}\right)\tan\left(\frac{\phi}{2}\right) \quad = \frac{1-\cos(\phi)}{\sin(\phi)} = \frac{\sin(\phi)}{1+\cos(\phi)}.$$

We can use our half angle formula to write

$$\begin{aligned} d_{\mathbb{H}}(P,Q) &= \ln\left(\frac{\sin(\theta)(1+\cos(\phi))}{\sin(\phi)(1+\cos(\theta))}\right) \\ &= \ln\left(\frac{\sin(\phi)(1-\cos(\theta)}{\sin(\theta)(1-\cos(\phi)}\right) \end{aligned}$$

Computing the hyperbolic cosines, we find

$$\begin{aligned} \cosh(|AC|) &= \frac{1}{2}\left(\frac{\sin(\theta)(1+\cos(\phi))}{\sin(\phi)(1+\cos(\theta))} + \frac{\sin(\theta)(1-\cos(\phi)}{\sin(\phi)(1-\cos(\theta)}\right) \\ &= \frac{1}{2}\frac{\sin(\theta)(2-2\cos(\theta)\cos(\phi)}{\sin(\phi)\sin^2(\theta)} \\ &= \frac{1-\cos(\theta)\cos(\phi)}{\sin(\theta)\sin(\phi)} \end{aligned}$$

The formula given is

$$\begin{aligned} 1+\frac{\|P-Q\|^2}{2y_P y_Q} &= 1+\frac{r^2\left((\cos(\theta)-\cos(\phi))^2+(\sin(\theta)-\sin(\phi))^2\right)}{2r^2\sin(\theta)\sin(\phi)} \\ &= 1+\frac{2-2\cos(\theta)\cos(\phi)-2\sin(\theta)\sin(\phi)}{2\sin(\theta)\sin(\phi)} \\ &= \frac{1-\cos(\theta)\cos(\phi)}{\sin(\theta)\sin(\phi)} \end{aligned}$$

as needed.

3. If d is the hyperbolic distance, we know that

$$\cosh(d) = 1+\frac{(2-1)^2+(7-1)^2}{2(1)(7)} = 1+\frac{37}{14} = \frac{51}{14}$$

Hence $d = \cosh^{-1}(51/14) \approx 1.9665$.