MAU22302/33302 - Euclidean and non-Euclidean Geometry

Tutorial Sheet 7

Trinity College Dublin

Course homepage

Exercise 1 *Stereographic distortion*

Let us consider some properties of projections for a sphere of radius 1. Recall that stereographic projection maps spherical circles not containing the north pole to Euclidean circles. Let σ be a spherical circle of spherical radius s whose centre is at angular distance ϕ from the north pole. Determine

1. The equation of the plane containing σ
2. The equation of the image of σ under stereographic projection
3. The ratio of the area of the image and the area of σ

as functions of s and ϕ.

Hint: Recall that the area of σ is $2\pi s^2(1 - \cos(s))$. The equation of the image of σ was given in lectures in terms of the equation of the plane.

Solution 1

1. Wlog, we can take the centre of our circle (rotating around the z-axis if necessary) to lie in the xz-plane and hence has coordinates

$$p = (\sin(\phi), 0, \cos(\phi))$$

A point $q \in \mathbb{S}^2$ is at angular distance s from p if

$$\langle q, p\rangle = \cos(s)$$

Thus the plane $\langle v, p\rangle = \cos(s)$ contains all points of σ and is the desired plane.

$$\sin(\phi)x + \cos(\phi)z = \cos(s)$$

We can alternatively note that the plane would be uniquely determined by the dot product with its normal vector being a constant. The normal vector is going to be p, and so it is just a matter of working out the constant by choosing some $q \in \sigma$ and a bit of Euclidean trigonometry.

2. In lectures, we worked out that the image of a spherical circle defined by the plane

$$Ax + By + Cz + D = 0$$

would be the circle

$$\left(u + \frac{A}{C+D}\right)^2 + \left(v + \frac{B}{C+D}\right)^2 = \frac{A^2 + B^2 + C^2 - D^2}{(C+D)^2}$$

assuming $C + D \neq 0$. We derived this using the explicit form for stereograph projection

$$\rho_S(x, y, z) = \left(\frac{-x}{1-z}, \frac{-y}{1-z}, -1\right)$$

and its inverse

$$\rho^{-1}(u, v) = \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right)$$

In our case, as we assume our circle (and hence the plane) does not contain $(0, 0, 1)$, we have that $C + D \neq 0$, so all is well.

3. The image is a circle of Euclidean radius

$$r^2 = \frac{A^2 + B^2 + C^2 - D^2}{(C+D)^2} = \frac{\sin^2(\phi) + \cos^2(\phi) - \cos^2(s)}{(\cos(\phi) - \cos(s))^2}$$

which simplifies to

$$r^2 = \frac{1 - \cos^2(s)}{(\cos(\phi) - \cos(s))^2}$$

Hence, the ratio of the areas is

$$\frac{\text{Area}(Image)}{\text{Area}(Circle)} = \frac{1 + \cos(s)}{2(\cos(\phi) - \cos(s))^2}$$

Exercise 2 *Length space metrics*

1. Let (X, ℓ, d_ℓ) be a length space. Show that the induced metric

$$d_\ell(p,q) = \inf_{\gamma:p\to q} \ell(\gamma)$$

is indeed a metric.

2. Let $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ be a norm. Show that, at least for differentiable

$$\gamma : [0,1] \to \mathbb{R}^2$$

the function

$$\ell(\gamma) := \int_0^1 \|\gamma'(t)\| \, dt$$

is a length function. That is, if $\gamma(t) = (x_0, y_0)$, then $\ell(\gamma) = 0$, and if

$$\gamma(t) = \begin{cases} \gamma_1(2t) \text{ if } 0 \le t \le \frac{1}{2}, \\ \gamma_2(2t-1) \text{ if } \frac{1}{2} \le t \le 1 \end{cases}$$

for γ_1, γ_2 paths such that $\gamma_1(t) = \gamma_2(0)$, then

$$\ell(\gamma) = \ell(\gamma_1) + \ell(\gamma_2).$$

Solution 2

1. As the set of paths from p to q is equal to the set of paths from q to p, we have $d(p,q) = d(q,p)$. We have that $d(p,q) = 0$ if and only if $p = q$, as $\ell(\gamma) = 0$ only if γ is the constant path, and (as X is a length space) there exists a $\gamma : p \to q$ such that

$$d(p,q) = \ell(\gamma).$$

Finally, for the triangle inequality, we need to do a bit of work. Let $p, q, r \in X$ and let $\gamma_{pq}, \gamma_{qr}, \gamma_{pr}$ be paths between them, such that $\ell(\gamma_{xy} = d(x,y)$. Then

$$d(p,r) = \int_{\gamma:p\to r} \ell(\gamma)$$

and so $d(p,r)$ is at most the length of any path from p to r passing through q, as all such paths appear in the infimum. In particular,

$d(p, q)$ is less than the length of the path obtained by concatenating γ_{pq} and γ_{qr}:

$$d(p, q) \leq \ell(\gamma_{qr}\gamma_{pq}) = \ell(\gamma_{qr}) + \ell(\gamma_{pq}) = d(q, r) + d(p, q).$$

Note that it is not necessary that X is a length space for the triangle inequality to hold! This holds in a path space. We only need to work in a length space to obtain that the metric separates distinct points. This could also be achieved by artificially imposing a lower bound on the length of paths between distinct points, which is not an uncommon solution.

2. As $\|\gamma'(t)\| \geq 0$, the integral is non-negative. Indeed, as $\|\gamma'(t)\| \geq 0$, the integral can only be 0 if $\|\gamma'(t)\| = 0$ for all $0 \leq t \leq 1$. This can only occur if the derivative vanishes, i.e. $\gamma(t)$ is constant.

   For the concatenation property, we have that, at least away from $t = \frac{1}{2}$,

$$\gamma'(t) = \begin{cases} 2\gamma_1'(t), \ 0 \leq t < \frac{1}{2}, \\ 2\gamma_2'(2t-1), \ \frac{1}{2} < t \leq 1 \end{cases}$$

   and so

$$\begin{aligned} \ell(\gamma) &= \int_0^1 \|\gamma'(t)\| \, dt \\ &= \int_0^{\frac{1}{2}} 2\|\gamma_1'(2t) \, dt + \int_{\frac{1}{2}}^1 2\|\gamma_2'(2t-1) \, dt. \end{aligned}$$

   Letting $u = 2t$ in the first integral and $u = 2t - 1$ in the second, we find

$$\begin{aligned} \ell(\gamma) &= \int_0^1 \|\gamma_1'(u)\| \, du + \int_0^1 \|\gamma_2'(u)\| \, du \\ &= \ell(\gamma_1) + \ell(\gamma_2). \end{aligned}$$

Exercise 3 *Optional: Polyhedral projection*

Consider a spherical projection defined as follows: construct a regular polyhedron P with faces tangent to $\mathbb{S}^2$, e.g. a cube or a tetrahedron. Define $\rho'(x)$ to be the intersection of the line from the origin through x with P. Fix a net/unfolding of P in the plane, and define $\rho_P(x)$ to be the image of $\rho'(x)$ under this unfolding.

1. This projection sees all points of the sphere, but is not single valued. What X would we have to remove from $\mathbb{S}^2$ to obtain a genuine map?

2. Describe the image of spherical lines under ρ_P.

3. Does ρ_P map spherical polygons to Euclidean polygons?

Hint: There is no need to be too formal here. Think about a cube and what that would look like.

Solution 3

1. In order to convert our polyhedron to a net, we need to choose a certain number of cut edges that are "repeated" in the plane. These are exactly the problem points. Let $S \subset P$ be the set of cut edges using in forming the net/unfolding. Then removing

$$X = (\rho')^{-1}(S)$$

from $\mathbb{S}^2$, we obtain a genuine single valued map

$$\rho_P : \mathbb{S}^2 \setminus X \to \mathbb{R}^2.$$

2. If F is a face of P, it is clear that $(\rho')^{-1}(F)$ is contained within an open hemisphere, and so, restricted to this inverse image, ρ_P agrees with gnomonic projection. Thus, the image of a spherical line under ρ_P in each face of the net is a Euclidean line. However, if the spherical line crosses $(\rho')^{-1})(E)$ for some edge E, then its image will either tilt as it crosses the image of E in the net (if E is not a cut edge), or will break into multiple disjoint line segments (if E is a cut edge). Thus, the image of a spherical line will be a union of possible disjoint Euclidean line segments, intersecting only at their endpoints.

3. Not in general. If the spherical polygon crosses the set X corresponding to cut edges, then we will obtain a collection of disjoin fragments of a Euclidean polygon's boundary. However, if the spherical polygon is contained entirely within $\mathbb{S}^2 \setminus X$, then we will obtain a Euclidean polygon, though with more vertices than our original polygon.

Exercise 4 *Optional: Norm induced lengths*

Define the taxicab norm by

$$\|(x, y)\| = |x| + |y|$$

1. Compute the length of the following curves in the induced length function.
   (a) $\gamma(t) = (t, t)$,
   (b) $\gamma(t) = (x(t), y(t))$ a differentiable function with $\gamma(0) = (0, 0)$, $\gamma(1) = (1, 1)$, and $x(t), y(t)$ monotonic.
2. Verify that the metric induced by the length is the metric induced by the norm. Does this hold more generally?

Solution 4

1. Let's compute some integrals.
   (a) We have that $\gamma'(t) = (1, 1)$ which has norm 2. Thus

   $$\ell(\gamma) = \int_0^1 2\,dt = 2.$$

   (b) As $x(t), y(t)$ are monotonic, their derivatives have constant sign. As such, we have that

   $$\|\gamma(t)\| = \pm x'(t) \pm y'(t)$$

   where the signs do not depend on t, and are chosen to ensure non-negativity

   $$\begin{aligned}\ell(\gamma) &= \pm \int_0^1 x'(t)\,dt + \pm \int_0^1 y'(t)\,dt \\ &= \pm(x(1) - x(0)) \pm (y(1) - y(0)) = \pm 1 \pm 1 = 2\end{aligned}$$

   as we take the non-negative value in both terms. Thus, we get that all monotonic paths between these points have equal length!

2. As in our last computation, we find that for the taxicab norm, all monotonic paths have equal length. Furthermore, it is clear to see that, for $\gamma(t)$ a monotonic path from p to q

$$\ell(\gamma) = \pm(x(1) - x(0)) \pm (y(1) - y(0)) = \|\gamma(1) - \gamma(0)\| = \|p - q\|$$

Thus, as long as this is a path of minimal length, the induced metric will agree with that induced by the norm.

To see that any non-monotonic path must have strictly greater length, we consider such a path γ. If we assume γ has continuous derivative, then we can find a finite partition

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

such that γ is monotonic on each $[t_i, t_{i+1}]$. Then, dividing up the integral accordingly, we find

$$\ell(\gamma) = \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\|$$

computed as before. Then, by the triangle inequality, we have that

$$\ell(\gamma) \geq \|\sum_{i=1}^{n} \gamma(t_i) - \gamma(t_{i-1})\| = \|\gamma(1) - \gamma(0)\| = \|p - q\|.$$

Thus, a monotonic path is a path of minimal length, and the induced metric is the metric induced by the norm.

Does this hold for more general norms? Yes! The straight line path

$$\gamma(t) = (1 - t)p + tq$$

has length

$$\ell(\gamma) = \int_0^1 \|q - p\| \, dt = \|p - q\|$$

for any norm $\| \cdot \|$, and we can show that this is minimal.

For sufficiently nice curves, we can define also their length by

$$L(\gamma) = \sup_P \sum_{i=1}^{n} \|\gamma(t_{i+1}) - \gamma(t_i)\|$$

where we take the supremum over partitions of $[0,1]$. Using the mean value theorem, we can rewrite this as

$$L(\gamma) = \sup_P \sum_{i=1}^{n} \|\gamma'(t_i^*)\||t_{i+1} - t_i| = \sup_P \sum_{i=1}^{n} \|\gamma'(t_i^*)\|\Delta t_i = \int_0^1 \|\gamma'(t)\, dt$$

as the norm is non-negative. Thus

$$\ell(\gamma) = \sup_P \sum_{i=1}^{n} \|\gamma(t_{i+1}) - \gamma(t_i)\|$$

for sufficiently nice curves. Hence, if we take the infimum over sufficiently nice curves

$$\begin{aligned}\ell(\gamma) = \sup_P \sum_{i=1}^{n} \|\gamma(t_{i+1}) - \gamma(t_i)\| \\ &\geq \sup_P \| \sum_{i=1}^{n} \gamma(t_{i+1}) - \gamma(t_i)\| \\ &= \sup_P \gamma(1) - \gamma(0)\| \\ &= \|p - q\|.\end{aligned}$$

and hence $d(p,q) \geq \|p-q\|$. Thus, we get equality.

As a final remark, we can use this $L(\gamma)$ function to define length for a more general metric space (X, d_X), and will still have that (for sufficiently nice curves)

$$\inf_{\gamma:p\to q} L(\gamma) \geq d_X(p,q)$$

but we usually cannot say that they will be equal without imposing conditions to make the metric space behave like $\mathbb{R}^n$, such as completeness and convexity.