

MAU22302/33302 - Euclidean and non-Euclidean Geometry

Tutorial Sheet 6

Trinity College Dublin

Course homepage

Exercise 1 *Spherical angles*

Recall that a spherical angle between two great circle segments is defined as the dihedral angle between the associated planes. We will derive a couple of formulae for this angle in terms of vector operations. You may assume we are working on a sphere of radius 1

1. Let $u, v, w \in \mathbb{S}_1^2$ be three distinct points, none of which are antipodal. Determine a formula for the cosine of the angle $\angle vuw$ in terms of cross and dot products

Hint: Rather than work with planes, work with their normal vectors

2. Show that this formula can be rewritten as

$$\cos(\theta) = \frac{\langle v, w \rangle - \langle u, v \rangle \langle u, w \rangle}{\|u \times v\| \|u \times w\|}$$

and hence express the angle in terms of the angular distances between u, v and w .

Solution 1

1. Sketching some spheres may be helpful in setting this up! We will also denote the inner product as a dot product $a \cdot b = \langle a, b \rangle$ in this answer.

A normal vector to the plane defining the arc uv is the cross product $u \times v$, which is non-zero as u and v are not antipodal. Similarly, a normal vector to the plane defining the arc uw is $u \times w$. The angle between the planes is equal to the angle between the normal vectors, and hence

$$\|u \times v\| \|u \times w\| \cos(\theta) = (u \times v) \cdot (u \times w).$$

2. This was an unfair problem, as it requires you to have seen and remember both the scalar triple product and vector triple product identities, or be committed to working them out. But, with these in mind, it is easy enough.

The scalar triple product identity says that

$$a \cdot (b \times c) = b \cdot (c \times a)$$

and so

$$(u \times v) \cdot (u \times w) = -(u \times v) \cdot (w \times u) = -w \cdot (u \times (u \times v)).$$

The vector triple product identity says that

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$$

and so

$$\begin{aligned} u \times (u \times v) &= (u \cdot v)u - (u \cdot u)v \\ &= (u \cdot v)u - v \end{aligned}$$

as u is a unit vector. Taking the dot product with $-w$ then gives the result.

Recalling that u, v, w are unit vectors, their angular distances are given by

$$\begin{aligned} d(v, w) &= \alpha = \cos^{-1}(v \cdot w) \\ d(u, w) &= \beta = \cos^{-1}(u \cdot w) \\ d(u, v) &= \gamma = \cos^{-1}(u \cdot v). \end{aligned}$$

As the norm of a cross product is given essentially by the sine of the angle in between the vectors

$$\|a \times b\| = \|a\|\|b\| \sin(\sigma)$$

we find

$$\cos(\theta) = \frac{\cos(\alpha) - \cos(\beta) \cos(\gamma)}{\sin(\beta) \sin(\gamma)}.$$

Exercise 2 *Spherical triangles and Euler characteristic*

1. Recall that a spherical triangle is called proper if its great circles divide the sphere into 8 regions. Show a spherical triangle T is proper if and only if all internal angles are less than π .
2. As in the Euclidean world, we can define the Euler characteristic of a spherical polygon. Unlike the Euclidean world, we can give a finite triangulation of the entire sphere, and hence assign an Euler characteristic to the sphere (without having to talk about limits)

By considering the areas of the triangles in a triangulation of the sphere, show that

$$V - E + F = 2$$

Hint: Area will give you are relationship between V, F and 2. To introduce E , try to count how many edges each triangle contributes and how many times each edge is contributed.

Solution 2

1. Suppose T is a proper spherical triangle. Then, at each vertex, its great circles divide a small disc into 4 angles, with opposite pairs being equal:

$$\theta_1, \theta_2, \theta_3 = \theta_1, \theta_4 = \theta_2$$

As these angles sum to 2π , we have

$$\theta_1 + \theta_2 = \pi$$

and in particular, both are less than π .

Now suppose T is not a proper spherical triangle. Then one of the triangles formed by its great circles intersects the interior of T . In particular, one of the great circles passes through the interior of T . Suppose T has vertices A, B, C and that the great circle defining the arc AB cuts through the interior of T . As such, the angle between the half-planes corresponding to AB and BC must be at least the angle between the half-plane corresponding to AB and the half-plane corresponding to the arc of the great circle cutting through T . But these half planes are part of the same plane, so that angle is π .

2. The area of a spherical triangle T is $r^2(\Theta_T - \pi)$ where Θ_T is the sum of the internal angles of T . Hence, given a triangulation of the sphere \mathcal{T} , we must have

$$4\pi r^2 = \sum_{T \in \mathcal{T}} r^2(\Theta_T - \pi) = r^2 \sum_{T \in \mathcal{T}} \Theta_T - \pi r^2 F$$

where F is the number of triangles (faces). In the sum of internal angle sums, we will sum over every internal angle in the triangulation so

$$\sum_{T \in \mathcal{T}} \Theta_T = \sum_{v \in V\mathcal{T}} \text{Sum of angles incident to } v$$

As the triangulation covers the sphere, the sum of angles incident to a vertex of the triangulation must be 2π (as we cover a full circle around the vertex). Then, we have

$$4\pi r^2 = 2\pi r^2 V - \pi r^2 F$$

and so

$$2 = V - \frac{1}{2}F.$$

To introduce E , we note that every triangle contributes 3 edges to the triangulation, but that each edge is counted twice in this way, so

$$2E = 3F, \quad \Rightarrow \quad E - F = \frac{1}{2}F$$

and so

$$2 = V - E + F.$$