# MAU22302/33302 - Euclidean and non-Euclidean Geometry

Tutorial Sheet 5

Trinity College Dublin

Course homepage

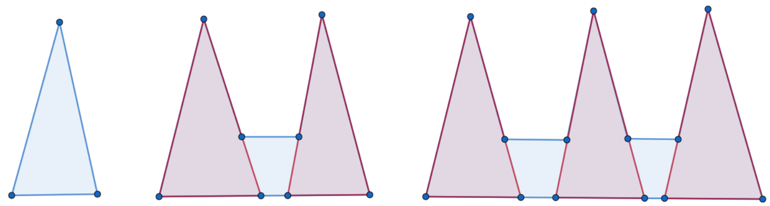
## Exercise 1 *The National Gallery is inefficient*

In lectures, we showed that a polygonal room with $n$ vertices could be effectively guarded by a group of $\lfloor \frac{n}{3} \rfloor$ guards. By explicitly constructing an example for every $n = 3m$, show that this is the minimal number of guards we can hire without more knowledge of the floor plan than the number of corners.

*Hint: The National Gallery has a room that's not too far off a worst case scenario. If that doesn't help, try a few small examples with have points that can only be seen from small areas, and glue those together so that guards views don't overlap too much*

## Solution 1

For $n = 3$, a triangle works. For $n = 6$, we can imagine two adjacent triangles. The apex of each triangle can only be seen by a guard in the triangle, so if we join the two triangles together with a short corridor, we can arrange it so that no point of the polygon is in both triangles. Similarly, we can add a third triangle via a short corridor so that no point of the polygon is in any pair of triangles, and so on. In this way, we construct a “comb” where a given guard can see at most one apex of the “teeth”. Thus, we need at least as many guards as there are teeth, which we can check to be exactly $\lfloor \frac{n}{3} \rfloor$



Here, the red areas represent the points that can see the apex of a given triangle, and the blue area is the "corridor". Taking the outer boundary gives the desired comb

## Exercise 2 *Small lattice polygons*

1. Given a lattice polygon $P$ (on $\mathbb{Z}^2$) of area 4, what are the possible values of $(I_P, B_P)$?
2. Does there exist a lattice polygon of area 4 with each of these interior/boundary vertex counts?

## Solution 2

1. By Pick's theorem, we must have $I_P + B_P/2 - 1 = 4$, or $2I_P + B_P = 10$. As $B_P \geq 3$ and both are integers, this means that $(I_P, B_P)$ is one of

$$(3, 4),\ (2, 6),\ (1, 8),\ (0; 10).$$

2. There are infinitely many lattice polygons of a given area, but an example for each $(I_P, B_P)$ pair is given here, found by squinting at it.

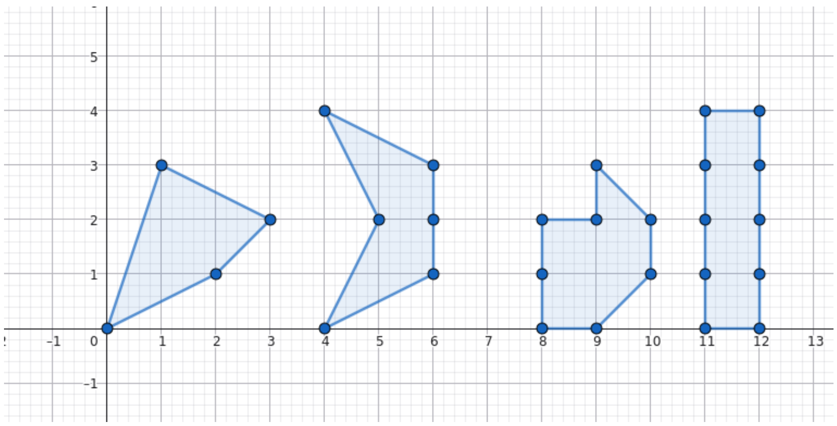
## Exercise 3 *Filling in the gaps*

We proved a generalisation of Pick's theorem to lattice polygons with holes by introducing the Euler characteristic and using this to count elementary triangles.

$$\text{Area}(P) = I_P + \frac{B_P}{2} - \chi(P)$$

where

$$\chi(P) = |V(P)| - |E(P)| + (1 - h(P))$$




A more direct proof would be via subtraction

1. By inducting on the number of holes, and using Pick's theorem for polygons, derive a formula for the area of a lattice polygon with holes in terms of the number of interior points, boundary points, and holes.

2. Hence, give an alternative explanation as to why $\chi_{\mathcal{T}}(P)$ depends only on $P$, at least for elementary triangulations $\mathcal{T}$.

## Solution 3

1. As $|V(P)| = |E(P)|$ for a polygon with no hole, the claim follows from Pick's theorem for $h(P) = 0$. Suppose the claim holds for polygons with $h-1$ holes, and let $P$ be a polygon with $h$ holes. Let $Q$ be one of the holes, and suppose the polygon $Q$ has $B_Q$ boundary points and $I_Q$ interior points. Let $\tilde{P}$ be the polygon with holes obtained from $P$ by ignoring $Q$. Then by induction on number of holes

$$\text{Area}(\tilde{P}) = I_{\tilde{P}} + \frac{B_{\tilde{P}}}{2} - \chi(\tilde{P})$$

and

$$\text{Area}(Q) = I_Q + \frac{B_Q}{2} - 1.$$

We want to compute

$$\text{Area}(P) = \text{Area}(\tilde{P}) - \text{Area}(Q) = I_{\tilde{P}} - I_Q + \frac{B_{\tilde{P}} - B_Q}{2} - \chi(\tilde{P}) + 1$$

We must have

$$I_P = I_{\tilde{P}} - I_Q - B_Q, \quad \text{and} \quad B_P = B_{\tilde{P}} + B_Q$$

and so

$$\begin{aligned}
\text{Area}(P) &= I_{\tilde{P}} - I_Q + \frac{B_{\tilde{P}} - B_Q}{2} - \chi(\tilde{P}) + 1 \\
&= I_{\tilde{P}} - I_Q - B_Q + B_Q + \frac{B_{\tilde{P}} - B_Q}{2} - \chi(\tilde{P}) + 1 \\
&= I_{\tilde{P}} - I_Q - B_Q + \frac{B_{\tilde{P}} + B_Q}{2} - \chi(\tilde{P}) + 1 \\
&= I_P + \frac{B_P}{2} - \chi(\tilde{P}) + 1.
\end{aligned}$$

Hence, it suffices to show that $\chi(P) = \chi(\tilde{P}) - 1$. We have that

$$\chi(\tilde{P}) = |V(\tilde{P})| - |E(\tilde{P}| + (1 - h(\tilde{P}))$$

It is clear that

$$|V(P)| = |V(\tilde{P})| + |V(Q)|, \quad \text{and} |E(P)| = |E(\tilde{P})| + |E(Q)| = |E(\tilde{P})| + |V(Q)|$$

and $h(\tilde{P}) = h(P) - 1$. Hence, the claim follows.

2. Our original proof of the generalisation of Pick's theorem works perfectly well replacing $\chi(P)$ with $\chi_{\mathcal{T}}(P)$ for an elementary triangulation $\mathcal{T}$. Indeed, the only time we used the independence of the Euler characteristic from the triangulation was to express the number of elementary triangles in terms of $\chi(P)$ rather than $\chi_{\mathcal{T}}(P)$. Hence, the same proof gives that

$$\text{Area}(P) = I_P + \frac{B_P}{2} - \chi_{\mathcal{T}}(P)$$

Comparing our two formulae, we must have that $\chi_{\mathcal{T}}(P) = \chi(P)$ for any elementary triangulation $\mathcal{T}$.