MAU22302/33302 - Euclidean and non-Euclidean Geometry

Tutorial Sheet 4

Trinity College Dublin

Course homepage

Exercise 1 *Reflection decompositions*

Let's decompose some examples of isometries of $\mathbb{R}^2$ into reflections.

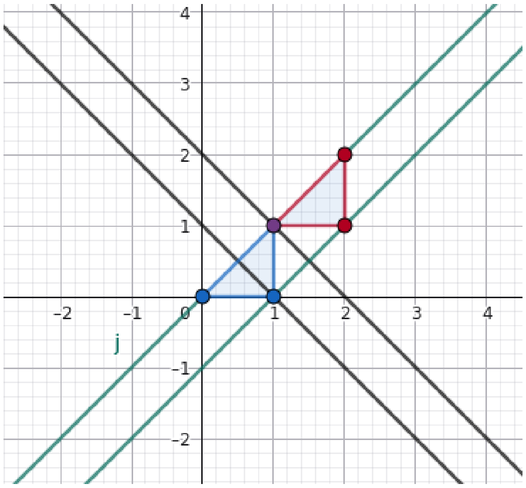
1. Consider the isometry $x \mapsto x + (1,1)$. Can you write this as a composition of two reflections? Can you find more than one pair of reflections whose composition is this translation?
2. Consider the rotation by $\pi/3$ anticlockwise around $(0,0)$. Can you write this as a composition of two reflections? Can you find more than one pair of reflections whose composition is this rotation?
3. Can you write the rotation by $\pi/3$ anticlockwise around $(1,1)$ as a composition of reflections? Can you do it with only two reflections?

Solution 1

1. It is enough to find a composition of two reflections that maps a triangle to the same triangle as the translation. I like the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$, but any triangle will do. To find the reflections, we consider first a reflection that will take $(0,0)$ to $(1,1)$, which will be given by the reflection r_1 in the line $y = 1 - x$ through both $(1,0)$

and $(0,1)$, as this line is the perpendicular bisector of the line segment $(0,0)-(1,1)$.

This leaves $(1,0)$ fixed and sends $(1,1)$ to $(0,0)$. Next we want to consider reflection in a line through $(1,1)$ (the image of $(0,0)$, taking either $(0,0)$ to $(1,1)+(1,1)=(2,2)$ or $(1,0)$ to $(1,0)+(1,1)=(2,1)$. The line $y=2-x$ is the perpendicular bisector of $(0,0)-(2,2)$ and contains $(1,1)$, so this is a candidate. Indeed, this reflection r_2 will map $(0,0)$ to $(2,2)$, and $(1,0)$ to $(2,1)$, as can be seen from the below diagram



Thus $r_2 \circ r_1$ is equal to the given translation.

Alternatively, we could first try to reflect $(1,1)$ to $(2,2)$ via the reflection R_1 in the line $y=\frac{3}{2}-x$. We have that

$$R_1(0,0)=(3,3), \quad R_1(1,0)=(3,2)$$

which can be worked out by careful drawing of squares. Then the reflection R_2 in the line $y=4-x$ fixes $(2,2)$, and

$$R_2(3,3)=(1,1), \quad R_2(3,2)=(2,1)$$

This $R_2 \circ R_1$ is also equal to the given translation.

2. Again, it suffices to consider only three points. As $(0,0)$ is fixed by the rotation, this is a convenient vertex for our triangle. The equilateral

triangle with base $(0,0)-(1,0)$ is a nice one to consider as rotation by $\pi/3$ will take $(1,0)$ to the third vertex.

If we consider reflections fixing $(0,0)$, and first try to move $(1,0)$ to the third vertex $(1/2, \sqrt{3}/2)$, we can achieve this via the reflection ρ_1 in the line through the origin at an angle of $\pi/6$. This fixes $(0,0)$ and swaps the other two vertices. If we then take the reflection ρ_2 in the line through the origin at an angle of $\pi/3$, this fixes the origin and the apex, and sends $(1,0)$ to $(-1/2, \sqrt{3}/2)$. Thus $\rho_2 \circ \rho_1$ is the desired rotation.

Alternatively, we could first try to move the vertex $(1/2, \sqrt{3}/2)$ to $(-1/2, \sqrt{3}2)$. We achieve this by reflection σ_1 in the y-axis. This fixes $(0,0)$ and sends $(1,0)$ to $(-1,0)$. The reflection σ_2 in the line through the origin at an angle of $2\pi/3$ then fixes $(0,0)$ and $(-1/2, \sqrt{3}2)$, and sends $(-1,0)$ to $(1/2, \sqrt{3}2)$, as desired. Thus $\sigma_2 \circ \sigma_1$ also gives the desired rotation.

3. The easiest way to write this rotation as a composition of reflections is to note that this is the composition of $x \mapsto x-(1,1)$, followed by rotation around the origin, followed by the translation $x \mapsto x+(1,1)$. Thus, the desired rotation can be given by

$$r_2 r_1 \rho_2 \rho_1 r_1^{-1} r_2^{-1}.$$

If we want only two rotations, we will again need to pick a nice triangle. Lets consider the equilateral triangle with vertices $(1,1)$, $(2,1)$, $(3/2, 1+\sqrt{3}/2)$, which will map to the triangle with vertices $(1,1)$, $(3/2, 1+\sqrt{3}/2)$, $(1/2, 1+\sqrt{3}/2)$. Similarly to previously, we can take our first reflection ζ_1 to be reflection in the line through $(1,1)$ at an angle of $\pi/6$ to the x-axis, and our second reflection ζ_2 to be reflection in the line through $(1,1)$ at an angle of π_3 to the x-axis.

Exercise 2 *Triangulations of polygons*

1. How many distinct strong triangulations are there of a regular n-gon for $n = 3,4,5,6$? What if we only consider strong triangulations distinct up to isometries (i.e. rotation and reflection)?

2. Show that for any polygon P with n vertices, the sum of the internal angles is $(n-2)\pi$.

Solution 2

1. There is one way to (strongly) triangulate a 3-gon, two ways to triangulate a 4-gon, 5 ways to triangulate a 5-gon, and 14 ways to triangulate a 6-gon. Up to isometry, there are $1, 1, 1, 3$ distinct ways to (strongly) triangulate a $3, 4, 5, 6$-gon respectively, soon to be shown below.

2. For any polygon with n vertices, there exists a strong triangulation into $n-2$ triangles. Each angle in one of these triangles is a piece of an internal angle of P, and all internal angles are completely covered by the angles of the triangles. Thus the sum of the internal angles is equal to the sum of the angles in the triangles, which is exactly $(n-2)\pi$.

Exercise 3 *If you have time for further reflection...*

We define reflection in $\mathbb{R}^3$ as follows. Suppose we have a plane P and a point x. If $x \in P$, then the reflection of x in P is x. Otherwise consider the normal line to P through x, intersection P at y. The reflection in P of x is then the unique other point $\tilde{x}$ on the normal line such that $d(x, y) = d(\tilde{x}, y)$. How could we show that reflection is an isometry? Can we reduce to the case of $\mathbb{R}^2$? Would this piggybacking let us show that reflection is an isometry in $\mathbb{R}^n$ for all $n \geq 2$?

Solution 3

We will show that, given $x_1, x_2 \in \mathbb{R}^3$, their images $r(x_1), r(x_2)$ under reflection in P are at the same distance by looking at 3 cases. If x_1 and x_2 are both contained in P, then they are not moved by the reflection and everything works fine.

If x_1 is in P, and x_2 is not (or vice versa), then either x_1 is the intersection of the normal line to P through x_2 with P, and the claim follows by definition, *or* the intersection point, x_1 and x_2 define a plane containing the normal line, and intersecting P in a line. The reflection of x_2 in P is then the (planar) reflection of x_2 in this intersection line. Hence, the distance will remain unchanged

If neither point is in P, we can again reduce to (planar) reflection in a line, assuming that x_1, x_2 and the normal-intersection points y_1, y_2 all lie in a plane. I claim the plane defined by x_1, y_1, y_2 contains x_2. I am positive there is a nicer way to do this, but we'll consider this approach for now.

We first note that we can translate and rotate so that $x_1 = (0, 0, K)$, $x_2 = (i, j, k)$, $y_1 = (0, 0, 0)$ and $y_2 = (i, j, 0)$. The plane containing x_1, y_1, y_2 is then given by the equation

$$ax + by + cz = d$$

for some constants a, b, c, d. Working out the simultaneous equations, we find that, up to rescaling the constants, the plane is given by

$$jx - iy = 0$$

Clearly, x_2 satisfies this. Thus, we can again reduce to the planar reflection case.