MAU22302/33302 - Euclidean and non-Euclidean Geometry

Tutorial Sheet 3

Trinity College Dublin

Course homepage

Exercise 1 *Plane isometries*

Let's ponder some examples of isometries of $\mathbb{R}^2$ with the standard Euclidean distance.

1. Does there exist an isometry mapping

$$(1,0) \mapsto \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \quad \text{and} \quad (0,1) \mapsto \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

   If so, find an example. You can describe it using words if easier.

2. Does there exist an isometry mapping

$$(1,0) \mapsto \left(1 + \frac{1}{\sqrt{2}}, \frac{3}{2}\right), \quad \text{and} \quad (0,1) \mapsto \left(1 - \frac{1}{\sqrt{2}}, \frac{3}{2}\right)$$

   If so, find an example. You can describe it using words if easier.

3. Let ℓ be an affine line in $\mathbb{R}^2$. Define a reflection map $\rho_\ell : \mathbb{R}^2 \to \mathbb{R}^2$ as follows. If x is a point on ℓ, then $\rho_\ell(x) = x$. Otherwise, there is a unique line ℓ_x through x perpendicular to ℓ, which intersects ℓ at p_x. We define $\rho_\ell(x)$ to be the unique other point on ℓ_x such that $d(x, p_x) = d(\rho_\ell(x), p_x)$. Is ρ_ℓ is an isometry?

   Hint: Go old-school on this. Be Greek about it

4. Suppose we have two lines ℓ_1 and ℓ_2. By considering how it acts on the vertices of a triangle, describe what kind of isometry $\rho_{\ell_2} \circ \rho_{\ell_1}$ could be.

   Hint: By rotating and translating the plane, you can assume ℓ_1 is the x-axis. By translating, you can assume the lines intersect in the origin (if they intersect at all). What might good choices for the triangle vertices be?

Solution 1

1. No! The distance between the two points and their images changes from $\sqrt{2}$ to something smaller!

2. Yes! The distance between the points is preserved, so we can find such an isometry. For example, we can take the composition of the anticlockwise rotation around the origin by $\frac{\pi}{4}$, which takes the line segment $(1,0)-(0,1)$ to one parallel to the desired image line segment, with the endpoints in the correct order, with the translation taking the leftmost endpoint of the rotated line segment to the leftmost endpoint of the image line segment. Explicitly, this is the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}.$$

3. We consider 3 cases, and suppress the subscript ℓ. First suppose both x and y are on ℓ. Then

$$d(\rho(x), \rho(y)) = d(x, y)$$

   by definition. If x is on ℓ, but y is not, then we consider the triangles $x - p_y - y$ and $x - p_y - \rho(y)$. By construction

$$d(x, p_y) = d(x, p_y),\ \angle x{-}p_y{-}y = \frac{\pi}{2} = \angle x{-}p_y{-}\rho(y),\ d(p_y, y) = d(p_y, \rho(y))$$

   so these triangles are equal/congruent. In particular $d(x, y) = d(x, \rho(y))$. Finally, if neither x nor y are on ℓ, then we consider first the triangles $p_x - p_y - y$ and $p_x - p_y - \rho(y)$. As before, these are congruent, and so $d(p_x, y) = d(p_x, \rho(y))$, and $\angle y - p_x - p_y = \angle \rho(y) - p_x - p_y$. As

$$\angle \rho(x) - p_x - p_y = \frac{\pi}{2} = \angle x - p_x - p_y$$

we therefore have $\angle \rho_x - p_x - \rho(y) = \angle x - p_x - y$. Finally $d(x, p_x) = d(\rho(x), p_x)$, so the triangles $x - p_x - y$ and the triangles $\rho(x) - p_x - \rho(y)$ are equal/congruent. Hence $d(x, y) = d(\rho(x), \rho(y))$.

4. We first consider the case of parallel lines. We can assume, by rotation and translation, that ℓ_1 is the x-axis, and that ℓ_2 is given by $y = c$ for some c. The composition is determined by the image of any three non-collinear points, so lets take $(0, 0)$, $(1, 0)$, and (a, b) for some a, b with $b \neq 0$. Reflection in ℓ_1 fixes $(0, 0)$ and $(1, 0)$, and sends (a, b) to $(a, -b)$. It is not hard to see that reflection in $y = c$ will send a point (x_0, y_0) to $(x_0, y_0 - 2(y - 0 - c)) = (x_0, 2c - y_0)$, so the composition of our reflections sends

$$(0, 0) \mapsto (0, 2c), \quad (1, 0) \mapsto (1, 2c), \quad (a, b) \mapsto (a, b + 2c)$$

i.e. translation by $(0, 2c)$. Thus composition of reflections in parallel lines gives a translation.

If the lines are not parallel, we can assume they intersect in the origin, that ℓ_1 is the x-axis and ℓ_2 is given by $y = mx$. We consider the points $(0, 0)$, $(1, 0)$, and $(1, -m)$, and we consider them as vertices of a triangle T We assume $(1, 0)$ is anti-clockwise of the edge $E : (0, 0) - (1, -m)$. The first reflection gives

$$(0, 0) \mapsto (0, 0), \quad (1, 0) \mapsto (1, 0), \quad (1, -m) \mapsto (1, m)$$

which gives a triangle T' congruent to T, but $(1, 0)$ is clockwise of the edge E'.

As $(0, 0)$, and $(1, m)$ are on ℓ_2, they are invariant under the reflection, so we just need to compute the image of $(1, 0)$. As reflection is an isometry, the image of $(1, 0)$ is the intersection of the circle of radius 1 centred at $(0, 0)$ and the circle of radius m centred at $(1, m)$ on the other side of ℓ_2 to $(1, 0)$. As E' lies along ℓ_2, $E'' = E'$. We again obtain a triangle T'' congruent to T, with the image of $(1, 0)$ anti-clockwise of the edge E''

If we considered a rotation of the plane so that the line $y = -mx$ is sent to the line $y = mx$, we would have a mapping such that

$$(0, 0) \mapsto (0, 0), \quad (1, -m) \mapsto (1, m)$$

so E maps to E''. The triangle T is mapped to a congruent triangle T_R, such that the vertex not on E'' is anticlockwise of E''. This is precisely the image of $(1, 0)$, and so the composition of the two reflections is a rotation around their intersection point.

We can make this more formal, by noting that the first reflection is given by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

while the second is given by

$$\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}$$

where $\tan(\theta) = m$. The product of these is precisely a rotation by 2θ.

Exercise 2 *Rotations in space*

Up to translation, a rotation in 3 dimensional space is given by a map of the form $x \mapsto Mx$, where M is a (3×3)-matrix such that $M^T M = I$ and $\det M = 1$.

1. Show that 1 is an eigenvalue of M and hence that M fixes a line through the origin. *Hint: consider $\langle Mv, Mv \rangle$ for some eigenvector v.*

2. Hence conclude that there exist a matrix P such that

$$P^{-1}MP = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $ad - bc = 1$.

Solution 2

1. Consider an eigenvector v of M, with eigenvalue λ. We have that

$$\langle Mv, Mv \rangle = \langle M^T Mv, v \rangle = \langle v, v \rangle$$

and

$$\begin{aligned}\langle Mv, Mv\rangle &= \langle \lambda v, \lambda v\rangle \\ &= |\lambda|^2\langle v, v\rangle.\end{aligned}$$

As $v \neq 0$, we have $|\lambda|^2 = 1$ for all eigenvalues. The eigenvalues of M satisfy a cubic polynomial

$$\det(M - xI) = 0$$

which either has 3 real roots, or 1 real root and two complex conjugate roots. Call the roots $\lambda_1, \lambda_2, \lambda_3$. As

$$\lambda_1\lambda_2\lambda_3 = \det(M) = 1, \quad \text{and} \quad |\lambda_i|^2 = 1$$

we have (up to relabelling) one of

$$\begin{cases} \lambda_1 = \lambda_2 = \lambda_3 = 1, \\ \lambda_1 = 1,\ \lambda_2 = \lambda_3 = -1, \\ \lambda_1,\ \lambda_2 = \overline{\lambda_3}. \end{cases}$$

In all cases, we have 1 as an eigenvalue.

Let w be the corresponding eigenvector. Then the like $\{tw \mid t \in \mathbb{R}\}$ is fixed by M.

2. Let w be an eigenvector with eigenvalue 1, of norm 1. We can find vectors u and v such that $\{u, v, w\}$ is an orthonormal basis (via Graham-Schmidt for example), and the matrix $P = [u\,v\,w]$ acts as an orthonormal change of basis matrix, so that $P^{-1}MP$ is the matrix whose first column is determined by Mu, whose second column is determined by Mv and whose third column is determined by Mw. We have

$$Mu = au + cv + ew,\ Mv = bu + dv + fw,\ Mw = w$$

for some a, b, c, d, e, f, so

$$P^{-1}MP = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ e & f & 1 \end{pmatrix}$$

By orthonormality, we have that

$$e = \langle w, Mu \rangle = \langle Mw, Mu \rangle = \langle M^T Mw, u \rangle = \langle w, u \rangle = 0$$

and similarly $f = 0$, as needed. The conditions on a, b, c, d follow from properties of the determinant and orthonormality

$$P^{-1} = P^T \;\Rightarrow\; (P^{-1}MP)(P^{-1}MP)^T = I, \quad \det(P^{-1}MP) = \det(M) = 1$$