MAU22302/33302 - Euclidean and non-Euclidean Geometry

Tutorial Sheet 2

Trinity College Dublin

Course homepage

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Circles in metric spaces*

We have largely brushed over circles in our discussion, as we can always define a circle in a metric space (X, d) by

$$C(x, r) = \{y \in X \mid d(x, y) = r$$

1. Sketch the unit circle $C((1, 0), 2)$ in $\mathbb{R}^2$ with the following metrics:

   (i) $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$

   (ii) $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

   (iii) $d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2, y_2|\}$

   (iv) $d_{IE}(x, y) = d_2(x, 0) + d_2(0, y)$ for all $x \neq y$.

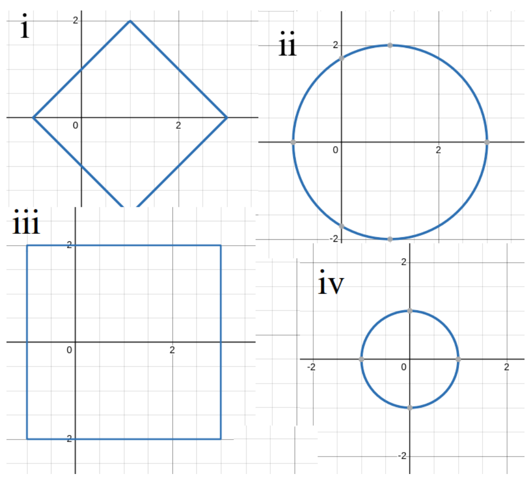
2. Recall that the metrics

$$d(x, y), \quad \text{and} \quad \tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

are considered equivalent as metrics, in that they capture the same notions of convergence. To further support that they define different geometries, try to sketch $C((1,1),1)$ for the metric $\tilde{d}$, where d is your favourite of the above metrics.

Solution 1

1. For parts i)-iv) we obtain a diamond, a circle, a square, and a circle centred at $(0,0)$ respectively. Part iv) is interesting, in that a "circle" can be a circle centred at $(0,0)$, a single point, or the empty set, depending on the radius compared to the d_2 distance between the centre and $(0,0)$.



2. The set is empty! If $\tilde{d}(x,y) = 1$, then $d(x,y) = 1 + d(x,y)$, which is impossible. There can be no unit circle in this metric!

Exercise 2 *Angles and collinearity in arbitrary metric spaces*

Using results from the lectures, we have seen how we can define angles and collinearity in Euclidean spaces using only the metric. We can arbitrarily extend these definitions to any metric space, but will it make sense?

1. Compute the cosine of the angle $\angle uvw$ for

$$u = (1,1), \quad v = (0,0), \quad w = (3,0)$$

in the metrics d_1, d_2, d_∞ from Exercise 1, using the formula for cosine in terms of distances.

2. Are these points collinear for any of these metrics?

3. Let $q = (1, -2)$. Compute cosine of the angles $\angle wvq$ and $\angle uvq$ in the given metrics. Does

$$\cos(a+b) = \frac{1}{2}\left(\cos(a)\cos(b) - \sin(a)\sin(b)\right)$$

   hold in these cases?

4. Let (X, d) be an arbitrary metric space. Is it true that $u, v, w \in X$ are collinear if and only if $\cos(\theta) = \pm 1$ where $\theta = \angle uvw$?

Solution 2

1. Recall that, for a metric d, the angle $\theta = \angle uvw$ is given by

$$\cos(\theta) = \frac{d(u,v)^2 + d(w,v)^2 - d(u,w)^2}{2d(u,v)d(w,v)}$$

   For the d_2 metric

$$\cos(\theta) = \frac{2+9-5}{2 \times \sqrt{2} \times 3} = \frac{1}{\sqrt{2}}$$

   For the d_1 metric

$$\cos(\theta) = \frac{4+9-9}{2 \times 2 \times 3} = \frac{1}{3}$$

   so the angle is "bigger" in this metric.

   For the d_∞ metric

$$\cos(\theta) = \frac{1+9-4}{2 \times 1 \times 3} = 1$$

   so the angle is zero in this metric!

2. Three points v_1, v_2, v_3 are collinear if and only if

$$d(v_1, v_3) = d(v_1, v_2) + d(v_2, v_3)$$

up to relabelling.

We compute

$$d_2(u, v) = \sqrt{2}, \quad d_2(v, w) = 3, \quad d_2(u, w) = \sqrt{5}$$

and it is clear that these three cannot be collinear in any order.

Similarly

$$d_1(u, v) = 2, \quad d_1(v, w) = 3, \quad d_1(u, w) = 3$$

and again, these cannot be collinear.

Finally

$$d_\infty(u, v) = 1, \quad d_\infty(v, w) = 3, \quad d_\infty(u, w) = 2.$$

As $1 + 2 = 3$, we conclude that u, w, v are collinear (in that order)

3. Let $\phi = \angle wvq$ and $\alpha = \angle uvq$.

For the d_2 metric, we compute

$$\cos(\phi) = \frac{1}{\sqrt{5}}, \quad \cos(\alpha) = \frac{-1}{2\sqrt{10}}.$$

It's not too difficult to compute

$$\sin(\theta) = \frac{1}{\sqrt{2}}, \quad \sin(\phi) = \frac{2}{\sqrt{5}}.$$

We see that

$$\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) = \frac{1}{\sqrt{10}} - \frac{2}{\sqrt{10}} = -\frac{1}{\sqrt{10}} = 2\cos(\alpha).$$

For the d_1 metric, we compute

$$\cos(\phi) = \frac{1}{9}, \quad \cos(\alpha) = \frac{1}{3}$$

and

$$\sin(\theta) = \frac{2\sqrt{2}}{3}, \quad \sin(\phi) = \frac{4\sqrt{5}}{9}.$$

We find

$$\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) = \frac{1}{27} - \frac{4\sqrt{5}}{81} \neq \frac{1}{3}.$$

Finally for the d_∞ metric, we compute

$$\cos(\phi) = \frac{3}{4}, \quad \cos(\alpha) = -1$$

and

$$\sin(\theta) = 0, \quad \sin(\phi) = \frac{\sqrt{7}}{4}.$$

We find

$$\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi) = \frac{3}{4} - 0 \neq -1.$$

The failure of the addition formula for d_1 and d_∞ angles relates to the fact that neither of these metrics come from inner products, despite coming from norms!

4. Suppose u, v, w are collinear. Then, up to renaming the points

$$d(u,v) + d(v,w) = d(u,w)$$

Hence

$$d(u,v)^2 + d(w,v)^2 + 2d(u,v)d(w,v) = d(u,w)^2$$

which we can rearrange to find

$$-1 = \frac{d(u,v)^2 + d(w,v)^2 - d(u,w)^2}{2d(u,v)d(w,v)} = \cos(\theta).$$

If $\cos(\theta) = -1$, then we can reverse the above argument to find u, v, w are collinear. If $\cos(\theta) = 1$, then we instead obtain

$$\left(d(u,v) - d(w,v)\right)^2 = d(u,w)^2.$$

Hence

$$d(u,v) - d(w,v) = d(u,w) \quad \Rightarrow d(u,v) = d(u,w) + d(w,v)$$

and so u, w, v are collinear.