MAU22302/33302 - Euclidean and non-Euclidean Geometry

Tutorial Sheet 1

Trinity College Dublin

Course homepage

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Euclid's assumptions and equilateral triangles*

As noted in lectures, Euclid's proof of the existence of equilateral triangles with given side length relies on the idea that the circles C_A and C_B centred at points A and B with shared radius $|AB|$ must intersect. Let's try to justify this.

(i) Give a definition of points *inside* a circle and points *outside* a circle.

(ii) Show that there is a point on C_A inside C_B and a point on C_A outside C_B, using only the postulates.

(iii) Hence, informally explain why C_A and C_B must intersect.

(iv) What kind of other assumptions would we need to make to make this explanation formal. Did we have to use any of them in the previous argument.

Solution 1

(i) Suppose we have a circle with centre O and radius r. Then P is inside the circle if $|PO| < r$, and outside the circle if $|PO| > r$. Note that this separates the plane into two disjoint regions - this is really only implicit in Euclid.

(ii) The point B is on C_A and is inside C_B, essentially by definition. We can extend the line AB through A to some point C such that $|AC| > |AB|$, i.e. a point outside the circle. The line $|AC|$ connects a point inside C_A to a point outside C_A and hence intersects C_A in a point D. Then

$$|BD| = |AB| + |AD| > |AB|$$

and so D is outside C_B.

(iii) Informally, any curve connecting a point inside C_B to a point outside C_B must intersect C_B. In particular, the circle C_A connects the point B (inside C_B) to a point outside C_B, and therefore C_A intersects C_B.

(iv) We have already had to make an assumption in part (ii)! Specifically, we need to know that a curve cannot "jump" between the inside and the outside. In modern formalisms, this would ask that these curves be continuous, and for us to know that path connected sets are connected.

Exercise 2 *Perpendiculars through a point*

(i) Let ℓ be a line containing distinct points A and B. Let P and Q be distinct points such that

$$|PA| = |PB|, \quad |QA| = |QB|.$$

Assume that PQ is not parallel to ℓ. Using the postulates and the first 5 propositions, show that the line PQ, suitably extended, is perpendicular to ℓ. (You may need to consider the cases of P, Q on the same/opposite sides of ℓ separately. Pictures may help!)

(ii) Hence, describe how to construct a line through a point P perpendicular to a line ℓ.

(iii) Can PQ ever be parallel to ℓ? Drop a perpendicular from each and compare the intersections. You may assume that the midpoint of a line segment is unique.

Hint: The last part really benefits from a diagram. It can be a bit hard to write down clearly.

Solution 2

(i) As APB and AQB are isosceles triangles, we have that

$$\angle PAB = \angle PBA, \quad \angle QAB = \angle QBA.$$

Let us first consider the case where P and Q lie on the same side of ℓ, and assume without loss of generality that $\angle PAB > \angle QAB$. Then

$$\angle PAQ = \angle PAB - \angle QAB = \angle PBA - \angle QBA = \angle PBQ.$$

Thus, $PAQ = PBQ$ as triangles. In particular $\angle APQ = \angle BPQ$.

If P and Q lie on opposite sides of ℓ, we reach the same conclusion, taking the sum of angles rather than the difference. In either case, extend PQ so that it intersects ℓ in a point R. We have

$$|AP| = |BP|, \quad \angle APR = \angle APQ = \angle BPQ = \angle BPR, \quad |PR| = |PR|$$

and so $APR = BPR$ as triangles. Hence

$$\angle PRA = \angle PRB.$$

But $\angle PRA + \angle PRB = \angle ARB = \pi$, and so both are right angles, as needed.

(ii) We construct a circle of centre P of radius large enough that it intersects ℓ in two points A and B. On the base AB, we construct an equilateral triangle ABQ, such that $Q \neq P$. (Why can we do this?) Then, by construction, we have distinct points P, Q such that

$$|PA| = |PB|, \quad |QA| = |QB|$$

and so PQ (appropriately extended) is perpendicular to ℓ.

(iii) There is clearly no issue if P and Q are on opposite sides of ℓ. Assume Q is on the same side of ℓ as P, and that PQ is parallel to ℓ. Draw a perpendicular line from P and Q to ℓ, intersecting ℓ at P' and Q' respectively. From the arguments part (i), we have that

$$PP'A = PP'B, \quad QQ'A = QQ'B$$

As triangles. In particular $|AP'| = |BP'|$ and $|AQ'| = |BQ'|$. Hence $P' = Q' = X$. As both PX and QX are perpendicular to ℓ, they must lie along the same line, which intersects the line PQ at a unique point. Hence PQ.

Exercise 3 *Inner products and norms*

Recall that an inner product on $\mathbb{R}^n$ is a bilinear map

$$I : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

such that

$$I(x, y) = I(y, x), \quad \text{and} \quad I(x, x) \geq 0$$

with equality if and only if $x = 0$. Let $\langle \cdot, \cdot \rangle$ denote the usual dot product.

(i) Show that there exists a matrix A such that

$$I(x, y) = \langle x, Ay \rangle$$

What are some properties of A?

(ii) Every inner product defines a norm by $\|x\|^2 = I(x, x)$. We showed that a norm comes from an inner product if and only if

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Does the norm

$$\|(x_1, \ldots, x_n)\| = |x_1| + |x_2| + \cdots + |x_n|$$

arise from an inner product?

Solution 3

(i) Let $e_1, \dots, e_n$ denote the standard basis of $\mathbb{R}^n$. By bilinearity

$$I(\sum_i x_i e_i, \sum_j y_j e_j) = \sum_{i,j} x_i y_j I(e_i, e_j).$$

For a matrix $A = A_{i,j}$, we have that

$$\begin{aligned}\langle x, Ay\rangle &= \langle \sum_i x_i e_i, \sum_j y_j A e_j\rangle \\ &= \sum_{i,j} x_i y_j \langle e_i, A e_j\rangle.\end{aligned}$$

Expanding out Ae_j in terms of the standard basis, we find $\langle e_i, Ae_j\rangle = A_{i,j}$. As such, defining the matrix A by $A_{i,j} = I(e_i, e_j)$, we find

$$I(x, y) = \langle x, Ay\rangle.$$

Symmetry of I implies that A is a symmetric matrix: $A_{i,j} = A_{j,i}$. Positive definiteness of I implies that $\langle x, Ax\rangle > 0$ for all $x \neq 0$, which implies A is positive definite. In fact, every positive definite real symmetric matrix defines an inner product.

(ii) Only if $n = 1$. Consider

$$x = (1, 0, \dots, 0), \quad y = (0, 1, 0, \dots, 0)$$

Then the LHS of the identity evaluates to 8, while the RHS evaluates to 4. If $n = 1$, then the LHS is

$$|x + y|^2 + |x - y|^2 = (x + y)^2 + (x - y)^2 = 2x^2 + 2y^2 = 2|x|^2 + 2|y|^2,$$