MAU22302/33302 - Euclidean and non-Euclidean Geometry

Tutorial Sheet 8

Trinity College Dublin

Course homepage

Exercise 1 *Hyperbolic reflection*

We saw that in Euclidean geometry that the composition of two reflections was either a translation (if the lines did not intersect) or a rotation (if they did intersect). Does this hold hyperbolically?

1. Consider two intersecting semicircles C_1, C_2 centred on the x-axis. Is the composition $\iota_{C_2} \circ \iota_{C_1}$ a Euclidean rotation? If not, can we interpret it as a hyperbolic rotation?

   Hint: In the Euclidean case, we would be rotating points along a circle centred at the point of intersection

2. Consider two non-intersecting semicircles C_1, C_2 centred on the x-axis. Is the composition $\iota_{C_2} \circ \iota_{C_1}$ a Euclidean translation?

3. In complex coordinates, determine a formula for the composition $\iota_{C_2} \circ \iota_{C_1}$ if C_1 is centred at the origin and has radius r, and C_2 is centred at $(a, 0)$ and has radius R.

4. In the Euclidean case we would be translating along a line perpendicular to both lines of reflection. Does such a line even exist in the hyperbolic case? Can we interpret the composition as translation along this line?

Hint: Consider tangent lines from the centre of this hyperbolic line. Is there necessarily a point where the tangent lines to each circle are equal?

Exercise 2 *Steps towards hyperbolic trigonometry*

Lets establish some simple trigonometry in $\mathbb{H}^2$

1. Suppose points $P, Q \in \mathbb{H}^2$ lie on a semicircle centred at the origin, with coordinates

$$P = (r\cos(\theta), r\sin(\theta)), \quad Q = (r\cos(\phi), r\sin(\phi)).$$

   Either by finding an appropriate isometry taking the hyperbolic line segment to a vertical line or by computing the length integral, show that

$$d_{\mathbb{H}}(P, Q) = \ln\left(\frac{\tan(\theta/2)}{\tan(\phi/2)}\right) = 2\tanh^{-1}\left(\frac{\sin((\theta-\phi)/2)}{\sin((\theta+\phi)/2)}\right)$$

   Hint: The semicircle containing P, Q is perpendicular to the x-axis. This is preserved under inversion. How can we get a line from inversion?

2. Show that

$$\cosh(d_{\mathbb{H}}(P, Q)) = 1 + \frac{\|P - Q\|^2}{2y_P y_Q}$$

   where $\|\cdot\|$ denotes the Euclidean distance, and y_X is the y-coordinate of the point X. Note that, as translation in the x-direction is a hyperbolic isometry, this gives a formula for all P, Q not on a vertical line (though it works for those too!).

   Hint: $\tan(\theta/2) = \frac{\sin(\theta)}{1+\cos(\theta)} = \frac{1-\cos(\theta)}{\sin(\theta)}$

3. Hence determine the hyperbolic distance between $(1, 1)$ and $(2, 7)$.

Remark 1. *Often, you see the hyperbolic distance expressed via the Euclidean norm*

$$d_{\mathbb{H}}(P, Q) = 2\tanh^{-1}\left(\frac{\|P - Q\|}{\|P - \tilde{Q}\|}\right)$$

where $\tilde{Q}$ is the reflection of Q in the x-axis. Drawing a diagram, using the cosine rule and the double angle formula, it is not hard to see this is equivalent to the formula we arrived at.