MAU22302/33302 - Euclidean and non-Euclidean Geometry

Tutorial Sheet 7

Trinity College Dublin

Course homepage

Exercise 1 *Stereographic distortion*

Let us consider some properties of projections for a sphere of radius 1. Recall that stereographic projection maps spherical circles not containing the north pole to Euclidean circles. Let σ be a spherical circle of spherical radius s whose centre is at angular distance ϕ from the north pole. Determine

1. The equation of the plane containing σ
2. The equation of the image of σ under stereographic projection
3. The ratio of the area of the image and the area of σ

as functions of s and ϕ.

Hint: Recall that the area of σ *is* $2\pi s^2(1-\cos(s))$. *The equation of the image of* σ *was given in lectures in terms of the equation of the plane.*

Exercise 2 *Length space metrics*

1. Let (X, ℓ, d_ℓ) be a length space. Show that the induced metric

$$d_\ell(p,q) = \inf_{\gamma:p\to q} \ell(\gamma)$$

   is indeed a metric.

2. Let $\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}$ be a norm. Show that, at least for differentiable

$$\gamma : [0,1] \to \mathbb{R}^2$$

   the function

$$\ell(\gamma) := \int_0^1 \|\gamma'(t)\|\, dt$$

   is a length function. That is, if $\gamma(t) = (x_0, y_0)$, then $\ell(\gamma) = 0$, and if

$$\gamma(t) = \begin{cases} \gamma_1(2t) \text{ if } 0 \le t \le \frac{1}{2}, \\ \gamma_2(2t-1) \text{ if } \frac{1}{2} \le t \le 1 \end{cases}$$

   for γ_1, γ_2 paths such that $\gamma_1(t) = \gamma_2(0)$, then

$$\ell(\gamma) = \ell(\gamma_1) + \ell(\gamma_2).$$

Exercise 3 *Optional: Polyhedral projection*

Consider a spherical projection defined as follows: construct a regular polyhedron P with faces tangent to $\mathbb{S}^2$, e.g. a cube or a tetrahedron. Define $\rho'(x)$ to be the intersection of the line from the origin through x with P. Fix a net/unfolding of P in the plane, and define $\rho_P(x)$ to be the image of $\rho'(x)$ under this unfolding.

1. This projection sees all points of the sphere, but is not single valued. What X would we have to remove from $\mathbb{S}^2$ to obtain a genuine map?

2. Describe the image of spherical lines under ρ_P.

3. Does ρ_P map spherical polygons to Euclidean polygons?

Hint: There is no need to be too formal here. Think about a cube and what that would look like.

Exercise 4 *Optional: Norm induced lengths*

Define the taxicab norm by

$$\|(x, y)\| = |x| + |y|$$

1. Compute the length of the following curves in the induced length function.

   (a) $\gamma(t) = (t, t)$,

   (b) $\gamma(t) = (x(t), y(t))$ a differentiable function with $\gamma(0) = (0, 0)$, $\gamma(1) = (1, 1)$, and $x(t), y(t)$ monotonic.

2. Verify that the metric induced by the length is the metric induced by the norm. Does this hold more generally?