MAU22302/33302 - Euclidean and non-Euclidean Geometry

Tutorial Sheet 6

Trinity College Dublin

Course homepage

Exercise 1 *Spherical angles*

Recall that a spherical angle between two great circle segments is defined as the dihedral angle between the associated planes. We will derivate a couple of formulae for this angle in terms of vector operations. You may assume we are working on a sphere of radius 1

1. Let $u, v, w \in \mathbb{S}^2_1$ be three distinct points, none of which are antipodal. Determine a formula for the cosine of the angle $\angle vuw$ in terms of cross and dot products

   Hint: Rather than work with planes, work with their normal vectors

2. Show that this formula can be rewritten as

$$\cos(\theta) = \frac{\langle v, w\rangle - \langle u, v\rangle\langle u, w\rangle}{\|u \times v\|\|u \times w\|}$$

   and hence express the angle in terms of the angular distances between u, v and w.

Exercise 2 *Spherical triangles and Euler characteristic*

1. Recall that a spherical triangle is called proper if its great circles divide the sphere into 8 regions. Show a spherical triangle T is proper if and only if all internal angles are less than π.

2. As in the Euclidean world, we can define the Euler characteristic of a spherical polygon. Unlike the Euclidean world, we can give a finite triangulation of the entire sphere, and hence assign an Euler characteristic to the sphere (without having to talk about limits)

   By considering the areas of the triangles in a triangulation of the sphere, show that
   $$V - E + F = 2$$

   Hint: Area will give you are relationship between V, F and 2. To introduce E, try to count how many edges each triangle contributes and how many times each edge is contributed.