MAU22302/33302 - Euclidean and non-Euclidean Geometry

Tutorial Sheet 3

Trinity College Dublin

Course homepage

Exercise 1 *Plane isometries*

Let's ponder some examples of isometries of $\mathbb{R}^2$ with the standard Euclidean distance.

1. Does there exist an isometry mapping

   $$(1,0) \mapsto \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right), \quad \text{and} \quad (0,1) \mapsto \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

   If so, find an example. You can describe it using words if easier.

2. Does there exist an isometry mapping

   $$(1,0) \mapsto \left(1 + \frac{1}{\sqrt{2}}, \frac{3}{2}\right), \quad \text{and} \quad (0,1) \mapsto \left(1 - \frac{1}{\sqrt{2}}, \frac{3}{2}\right)$$

   If so, find an example. You can describe it using words if easier.

3. Let ℓ be an affine line in $\mathbb{R}^2$. Define a reflection map $\rho_\ell : \mathbb{R}^2 \to \mathbb{R}^2$ as follows. If x is a point on ℓ, then $\rho_\ell(x) = x$. Otherwise, there is a unique line ℓ_x through x perpendicular to ℓ, which intersects ℓ at p_x. We define $\rho_\ell(x)$ to be the unique other point on ℓ_x such that $d(x, p_x) = d(\rho_\ell(x), p_x)$. Is ρ_ℓ is an isometry?

   Hint: Go old-school on this. Be Greek about it

4. Suppose we have two lines ℓ_1 and ℓ_2. By considering how it acts on the vertices of a triangle, describe what kind of isometry $\rho_{\ell_2} \circ \rho_{\ell_1}$ could be.

   Hint: By rotating and translating the plane, you can assume ℓ_1 is the x-axis. By translating, you can assume the lines intersect in the origin (if they intersect at all). What might good choices for the triangle vertices be?

Exercise 2 *Rotations in space*

Up to translation, a rotation in 3 dimensional space is given by a map of the form $x \mapsto Mx$, where M is a (3×3)-matrix such that $M^T M = I$ and $\det M = 1$.

1. Show that 1 is an eigenvalue of M and hence that M fixes a line through the origin

2. Hence conclude that there exist a matrix P such that

$$PMP^{-1} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

   and $ad - bc = 1$.