# MAU22302/33302 - Euclidean and non-Euclidean Geometry

Tutorial Sheet 2

Trinity College Dublin

Course homepage

The use of electronic calculators and computer algebra software is allowed.

## Exercise 1 *Circles in metric spaces*

We have largely brushed over circles in our discussion, as we can always define a circle in a metric space $(X, d)$ by

$$C(x, r) = \{y \in X \mid d(x, y) = r$$

1. Sketch the unit circle $C((1, 0), 2)$ in $\mathbb{R}^2$ with the following metrics:

   (i) $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$

   (ii) $d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$

   (iii) $d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2, y_2|\}$

   (iv) $d_{IE}(x, y) = d_2(x, 0) + d_2(0, y)$ for all $x \neq y$.

2. Recall that the metrics

$$d(x, y), \quad \text{and} \quad \tilde{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}$$

are considered equivalent as metrics, in that they capture the same notions of convergence. To further support that they define different geometries, try to sketch $C((1,1),1)$ for the metric $\tilde{d}$, where $d$ is your favourite of the above metrics.

## **Exercise 2** *Angles and collinearity in arbitrary metric spaces*

Using results from the lectures, we have seen how we can define angles and collinearity in Euclidean spaces using only the metric. We can arbitrarily extend these definitions to any metric space, but will it make sense?

1. Compute the cosine of the angle $\angle uvw$ for

$$u = (1,1), \quad v = (0,0), \quad w = (3,0)$$

   in the metrics $d_1, d_2, d_\infty$ from Exercise 1, using the formula for cosine in terms of distances.

2. Are these points collinear for any of these metrics?

3. Let $q = (1,-2)$. Compute cosine of the angles $\angle wvq$ and $\angle uvq$ in the given metrics. Does

$$\cos(a+b) = \frac{1}{2}(\cos(a)\cos(b) - \sin(a)\sin(b))$$

   hold in these cases?

4. Let $(X,d)$ be an arbitrary metric space. Is it true that $u, v, w \in X$ are collinear if and only if $\cos(\theta) = \pm 1$ where $\theta = \angle uvw$?