MAU22302/33302 - Euclidean and non-Euclidean Geometry

Tutorial Sheet 1

Trinity College Dublin

Course homepage

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Euclid's assumptions and equilateral triangles*

As noted in lectures, Euclid's proof of the existence of equilateral triangles with given side length relies on the idea that the circles C_A and C_B centred at points A and B with shared radius $|AB|$ must intersect. Let's try to justify this.

(i) Give a definition of points *inside* a circle and points *outside* a circle.

(ii) Show that there is a point on C_A inside C_B and a point on C_A outside C_B, using only the postulates.

(iii) Hence, informally explain why C_A and C_B must intersect.

(iv) What kind of other assumptions would we need to make to make this explanation formal. Did we have to use any of them in the previous argument.

Exercise 2 *Perpendiculars through a point*

(i) Let ℓ be a line containing distinct points A and B. Let P and Q be distinct points such that

$$|PA| = |PB|, \quad |QA| = |QB|.$$

Assume that PQ is not parallel to ℓ. Using the postulates and the first 5 propositions, show that the line PQ, suitably extended, is perpendicular to ℓ. (You may need to consider the cases of P, Q on the same/opposite sides of ℓ separately. Pictures may help!)

(ii) Hence, describe how to construct a line through a point P perpendicular to a line ℓ.

(iii) Can PQ ever be parallel to ℓ? Drop a perpendicular from each and compare the intersections. You may assume that the midpoint of a line segment is unique.

Hint: The last part really benefits from a diagram. It can be a bit hard to write down clearly.

Exercise 3 *Inner products and norms*

Recall that an inner product on $\mathbb{R}^n$ is a bilinear map

$$I : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$$

such that

$$I(x, y) = I(y, x), \quad \text{and} \quad I(x, x) \geq 0$$

with equality if and only if $x = 0$. Let $\langle \cdot, \cdot \rangle$ denote the usual dot product.

(i) Show that there exists a matrix A such that

$$I(x, y) = \langle x, Ay \rangle$$

What are some properties of A?

(ii) Every inner product defines a norm by $\|x\|^2 = I(x, x)$. We showed that a norm comes from an inner product if and only if

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Does the norm

$$\|(x_1, \ldots, x_n)\| = |x_1| + |x_2| + \cdots + |x_n|$$

arise from an inner product?