MAU22302/33302 - Euclidean and non-Euclidean Geometry

Homework 4

Trinity College Dublin

Course homepage Answers are due for April 8^{th}, 23:59.

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Möbius transformations*

i) Determine the image of $z = 1+2i$ (in the form $a+bi$) under the Möbius transformations given by the matrices

$$\gamma_1 = \begin{pmatrix} 4 & 3 \\ 0 & 2 \end{pmatrix}, \ \gamma_2 = \begin{pmatrix} 0 & 1 \\ -2 & 8 \end{pmatrix}, \ \gamma_3 = \begin{pmatrix} 1 & 1 \\ 2 & 7 \end{pmatrix}.$$

ii) Determine a real matrix γ with determinant 1 such that the corresponding Möbius transformation satisfies

$$\gamma \cdot i = \frac{8+i}{5}, \ \gamma \cdot (2i) = \frac{7+i}{4},$$

Hint: Reduce to linear equations in the matrix coefficients and rescale

iii) Show that if $\gamma, \delta \in \mathrm{SL}_2(\mathbb{R})$, then $\gamma \cdot (\delta \cdot z) = (\gamma\delta) \cdot z$, so that composition of Möbius transformations is compatible with matrix multiplication.

iv) Show that $\gamma \cdot z \in \mathbb{H}^2$ if $z \in \mathbb{H}^2$, for every $\gamma \in \mathrm{SL}_2(\mathbb{R})$, i.e. the action of the special linear group preserves the upper half plane.

Solution 1

i) We compute these as follows:

$$\begin{aligned}\gamma_1(z) &= \frac{4z+3}{0z+2} = 2z + \frac{3}{2} = \frac{7}{2} + 4i,\\ \gamma_2(z) &= \frac{0z+1}{-2z+8} = \frac{1}{6-4i}\\ &= \frac{6+4i}{36+16} = \frac{3}{26} + \frac{1}{13}i,\\ \gamma_3(z) &= \frac{z+1}{2z+7} = \frac{2+2i}{9+4i}\\ &= \frac{(2+2i)(9-4i)}{81+16} = \frac{26+11i}{97}\end{aligned}$$

ii) Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ so that

$$\gamma(z) = \frac{az+b}{cz+d}$$

Then we need

$$\frac{ai+b}{ci+d} = \frac{8+i}{5} \Leftrightarrow ai+b = \left(\frac{8+i}{5}\right)(ci+d) = \frac{8d-c}{5} + \frac{d+8c}{5}i$$

and so

$$5a = 8c+d, \quad 5b = 8d-c.$$

Similarly

$$\frac{b+2ai}{d+2ci} = \frac{7+i}{4} \Leftrightarrow b+2ai = \frac{7d-2c}{4} + \frac{14c+d}{4}i$$

so

$$4b = 7d-2c, \quad 8a = 14c+d.$$

Solving these equations in terms of d, we find

$$a = d,\ b = \frac{3d}{2},\ c = \frac{d}{2}$$

To ensure we get determinant 1, we need

$$1 = ad - bc = d^2 - \frac{3d^2}{4}, \quad \Leftarrow \quad d = \pm 2$$

Thus $\gamma = \left(\begin{smallmatrix} 2 & 3 \\ 1 & 2 \end{smallmatrix}\right)$ is such a matrix, as is the negative.

iii) If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $\delta = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, then

$$\gamma\delta = \begin{pmatrix} aA + bC & aB + bD \\ cA + dC & cB + dD \end{pmatrix}$$

Hence

$$(\gamma\delta)(z) = \frac{aAz + bCz + aB + bD}{cAz + dCz + cB + dD}.$$

We also have that

$$\delta(z) = \frac{Az + B}{Cz + D}$$

and so

$$\begin{aligned} \gamma(\delta(z)) &= \frac{a\frac{Az+B}{Cz+D} + b}{c\frac{Az+B}{Cz+D} + d} \\ &= \frac{aAz + aB + bCz + bD}{cAz + cB + dCz + cD} = (\gamma\delta)(z). \end{aligned}$$

iv) If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then

$$\gamma(z) = \frac{az + b}{cz + d} = \frac{(az + b)(c\overline{z} + d}{|cz + d|^2}$$

The denominator is positive, so it is enough to work out the numerator. If $z = x + iy$, then

$$(az+b)(c\overline{z}+d) = (ax+b+iay)(cx+d-icy) = (acx^2+(ad+bc)x+bd+acy^2)+(ad-bc)yi$$

As $ad - bc = 1$ for all $\gamma \in \mathrm{SL}_2(\mathbb{R})$, we find that $\gamma(z)$ has positive imaginary part and so is an element of $\mathbb{H}^2$.

Exercise 2 *Computing integrals trigonometrically*

Let $P = (\cos(b), \sin(b))$ and $Q = (\cos(a), \sin(a)$ be points on the semicircle σ of radius 1 centred at the origin, with $0 < a < \frac{\pi}{2} < b < \pi$, and let $\ell = (0, y)$ be the y-axis.

i) Determine a circle C centred on the x-axis such that inversion in C maps P and Q to points on ℓ

   Hint: When does inversion send circles to line?

ii) Using Euclidean trigonometry, determine the images $\iota_C(P)$ and $\iota_C(Q)$.

Hint: What lines must the images lie on? How could we find the angles those lines make with the x-axis?

iii) Hence determine $d_{\mathbb{H}}(P,Q)$ and conclude the identity

$$\tanh^{-1}(\cos(a)) - \tanh^{-1}(\cos(b)) = \ln\left(\frac{\tan\left(\frac{b}{2}\right)}{\tan\left(\frac{a}{2}\right)}\right)$$

Hint: You may freely use that $\int \frac{dx}{\sin(x)} = -\tanh^{-1}(\cos(x)) +$ *constant. This problem also gives an incredibly roundabout way of proving a half angle formula!*

Solution 2

i) Let $R = (1,0)$ be an intersection of σ and the x-axis, and let $M = (0,1)$ be the intersection of σ with ℓ. Let C be the circle with centre R through M. As σ contains R, inversion in C maps this circle to a Euclidean line. As σ is perpendicular to the x-axis (which is invariant under inversion in C), the image is a Euclidean line perpendicular to the x-axis. As $M \in \sigma$, σ is sent to a line containing its image. As $M \in C$, M is fixed by inversion in C. Putting these together, $\iota_C(\sigma)$ is a vertical Euclidean line through M, i.e. ℓ.

ii) As $\iota_C(P)$ is on ℓ and on the line from R to P, we have $P' = \iota_C(P)$ is the intersection of these lines. To compute the y-coordinate of P', we note that

$$|OP'| = \frac{|OP'|}{|OR|} = \tan(\angle ORP)$$

As $\triangle ORP$ is isosceles, we compute this angle to be $\frac{\pi - b}{2}$. Thus

$$|OP'| = \tan(\frac{\pi - b}{2}) = \cot(b/2).$$

Similarly, $|OQ'| = \cot(a/2)$. Thus

$$\iota_C(P) = (0, \cot(b/2)), \quad \iota_C(Q) = (0, \cot(a/2)).$$

iii) As inversion is an isometry

$$d_{\mathbb{H}}(P,Q) = d_{\mathbb{H}}(P',Q') = \ln\left(\frac{\cot(a/2)}{\cot(b/2)}\right) = \ln\frac{\tan(b/2)}{\tan(a/2)}.$$

To conclude the identity, we compute $d_{\mathbb{H}}(P,Q)$ using the integral formula. The hyperbolic line from Q to P is a segment of the unit circle, parametrised by

$$\{(\cos(t),\sin(t)) \mid a \le t \le b\}$$

This has hyperbolic length

$$\int_a^b \frac{\sqrt{(-\sin(t))^2 + (\cos(t))^2}}{\sin(t)}\,dt \;=\; \int_a^b \frac{dt}{\sin(t)}$$

Using the hint, we obtain exactly the given identity.

This gives a (very roundabout) was to prove a half angle formula. We can show the integral by multiplying by $\frac{\sin(t)}{\sin(t)}$. We can do some algebra to find

$$\tanh^{-1}(t) = \frac{1}{2}\ln\left(\frac{1+t}{1-t}\right).$$

Taking $a = \frac{\pi}{2}$, the corresponding inverse tanh term vanishes, and the corresponding tan term is equal to 1. Thus the identity gives

$$-\frac{1}{2}\ln\left(\frac{1+\cos(b)}{1-\cos(b)}\}\right) = \ln(\tan(b/2)).$$

and hence

$$\tan(b/2) = \sqrt{\frac{1-\cos(b)}{1+\cos(b)}}$$

at least for $\frac{\pi}{2} < b < \pi$. Squaring the identity, we obtain something that holds for all values of b.

Exercise 3 *Optional: Hyperbolic curvature*

Recall that for a geometry in which a circle with centre p and radius s has circumference $\mathrm{Circ}(C_p(s))$, we defined the Gaussian curvature at a point to be

$$K(p) = \lim_{s \to 0^+} 3\frac{2\pi s - \mathrm{Circ}(C_p(s))}{\pi s^3}$$

Let's determine the curvature of $\mathbb{H}^2$.

i) Let ABC be a hyperbolic triangle with $|AB| = |AC| = s$ (hyperbolically) and hyperbolic angle $\angle ABC = \alpha$, determine the hyperbolic length of BC.

   Hint: Think back to tutorial 8!

ii) By approximating the circle via a hyperbolic n-gon, hence compute $\mathrm{Circ}(C)$ as a function of s (and possibly (A, B)).

   Hint: What is $\lim_{n\to\infty} \sinh(an)/n$*?*

iii) Using whatever calculus is necessary, determine $K((A, B))$ for $\mathbb{H}^2$

   Having hopefully found that $\mathbb{H}^2$ has fundamentally different curvature to Euclidean space, lets consider whether we can still approximate hyperbolic circles well by Euclidean circles for very small or large radii

iv) Determine $\lim \frac{\mathrm{Circ}(C)}{2\pi s}$ as s tends to 0 or ∞.

v) If either limit tends to 1, this suggests the hyperbolic circle with small/large radius is very close to a Euclidean circle with the same centre and radius. But a hyperbolic circle *is* a Euclidean circle, with a different centre and radius. In the appropriate limiting case, do these centres and radii get close?

vi) If feeling incredibly brave, explicitly parametrise the hyperbolic circle and determine its circumference via our integral expression for hyperbolic length. The integral you get can either be evaluated using the Residue Theorem, or using the substitution $u = \tan\left(\frac{x}{2}\right)$ and some care around discontinuities

Solution 3

i) A hyperbolic circle with centre (A, B) and radius s is a Euclidean circle with centre $(A, B\cosh(s))$ and radius $B\sinh(s)$. Thus, a parametrisation is given by

$$x(t) = A + B\sinh(s)\cos(t), \quad y(t) = B\cosh(s) + B\sinh(s)\sin(t)$$

with $0 \leq t \leq 2\pi$, though a better choice of parametrisation will align $t = 0$ with a vertical line, so we will instead take

$$x(t) = A + B\sinh(s)\sin(t), \quad y(t) = B\cosh(s) + B\sinh(s)\cos(t).$$

We compute

$$x'(t) = B\sinh(s)\cos(t), \quad y'(t) = -B\sinh(s)\sin(t)$$

and so

$$(x'(t))^2 + (y'(t))^2 = (B\sinh(s))^2.$$

Hence the circumference is given by the integral

$$\mathrm{Circ}(C) = \int_0^{2\pi} \frac{B\sinh(s))}{B\cosh(s) + B\sinh(s)\cos(t)}\,dt = \int_0^{2\pi} \frac{dt}{\coth(s) + \cos(t)}$$

We have two way to compute this. The first is to consider this as a complete integral

$$\begin{aligned}\mathrm{Circ}(C) &= \int_0^{2\pi} \frac{2dt}{2\coth(s) + e^{it} + e^{-it}} \\ &= 2\int_0^{2\pi} \frac{e^{it}\,dt}{e^{2it} + 2\coth(s)e^{it} + 1} \\ &= \frac{2}{i}\int_{|z|=1} \frac{dz}{z^2 + 2\coth(s)z + 1}\end{aligned}$$

Let α, β be the roots of the denominator. As $\coth(s) > 1$ for all $s \geq 0$, exactly one of these lies within the unit circle, wlog α. Then

$$\frac{1}{z^2 + 2\coth(s)z + 1} = \frac{1}{\alpha - \beta}\left(\frac{1}{z - \alpha} - \frac{1}{z - \beta}\right)$$

Thus, by the residue theorem

$$\mathrm{Circ}(C) = \frac{2}{i}(2\pi i)\frac{1}{\alpha-\beta} = \frac{4\pi}{\alpha-\beta}$$

The roots of the polynomial are

$$-\coth(s)\pm\sqrt{\coth^2(s)-1} = -\coth(s)\pm\sqrt{\mathrm{cosech}^2(s)} = -\coth(s)\pm\mathrm{cosech}(s)$$

and so

$$\alpha-\beta = \pm 2\,\mathrm{cosech}(s) = \frac{\pm 2}{\sinh(s)}$$

As $-\coth(s)-\sqrt{\coth^2(s)-1} < -1$, this is the root outside the unit circle, so we have

$$\frac{\alpha-\beta}{=}\frac{\sinh(s)}{2}$$

and thus $\mathrm{Circ}(C) = 2\pi\sinh(s)$.

The second is to make the substitution $u = \tan(\frac{t}{2})$. As $\tan(x)$ is not defined at $\frac{\pi}{2}$, it makes sense to use that reflection in the line $x = A$ is an isometry, so that

$$\mathrm{Circ}(C) = 2\int_0^{\pi}\frac{dt}{\coth(s)+\cos(t)}$$

If $u = \tan(\frac{t}{2})$, then

$$du = \frac{1}{2\cos^2(t/2)} = \frac{1+u^2}{2}dt, \quad ,\cos(t) = 2\cos^2(t/2)-1 = \frac{1-u^2}{1+u^2}$$

Thus

$$\mathrm{Circ}(C) = 4\int_0^{\infty}\frac{du}{(A+1)+(A-1)u^2} = \frac{4}{\sqrt{A^2-1}}\tan^{-1}\left(\sqrt{\frac{A-1}{A+1}}t\right)\Big|_0^{\infty}$$

where $A = \coth(s)$. Evaluating the limits, we find

$$\mathrm{Circ}(C) = \frac{2\pi}{\sqrt{A^2-1}} = 2\pi\sinh(s).$$

Exercise 4 *Optional - Lengths in convex metric spaces*

For (X, d) a metric space, we can assign a length to any sufficiently nice curve $\gamma : [0, 1] \to X$ as follows

$$L_d(\gamma) := \sup_P \sum_{i=1}^{n} d(\gamma(t_{i+1}), \gamma(t_i))$$

where we take the supremum over partition of $[0, 1]$. This length could be infinite, but modulo this, serves as a good length function for sufficiently nice γ. From this, we can define a new metric

$$d_L(p, q) = \inf_{\gamma : p \to q} L_d(\gamma)$$

where we take the infimum over sufficiently nice paths from p to q. Using the triangle inequality, it is not hard to show

$$d_L(p, q) \geq d(p, q)$$

but when is it equal? We claim this holds if X is complete and convex.

We say a metric space X is convex if, for every $p, q \in X$ there exists $r \in X$ such that

$$d(p, q) = d(p, r) + d(r, q)$$

We will call X a midpoint metric space if such an r exists with $d(p, r) = d(r, q)$.

i) Let $C = d(p, q)$. Show that if X is a complete midpoint metric space, there exists a curve $\gamma : [0, 1] \to X$ such that

$$\gamma(0) = p, \quad \gamma(1) = q$$
$$d(p, \gamma(t)) + d(\gamma(t), q) = d(p, q) \text{ for all } 0 \leq t \leq 1$$
$$d(p, \gamma(t)) = Ct.$$

ii) Show that such a curve γ has $L_d(\gamma) = C$.

iii) Hence conclude that $d_L = d$ in a complete midpoint metric space.

iv) Does the same hold in a complete convex metric space? (For sake of your sanity, do not try to be rigorous about this one)

Solution 4

i) As X is a midpoint space, we can find an r such that

$$d(p,r) = d(r,q) = \frac{1}{2}d(p,q).$$

We can similarly find a midpoint of p and r, and a midpoint of r and q. In fact, by taking an appropriate sequence of midpoints, we can find a point $r_{a,n}$ such that

$$d(p, r_{a,n}) = \frac{a}{2^n}d(p,q) = \frac{Ca}{2^n}$$

for every $n \geq 0$, and $0 \leq a \leq 2^n$. The set of such points is dense in $[0,1]$, so for any $t \in [0,1]$, we can find a sequence (a_m, n_m) such that

$$\lim_{m\to\infty} \frac{a_m}{2^{n_m}} \to t$$

By construction, the sequence $\{r_{a_m,n_m}\}$ is a Cauchy sequence. As X is complete, it converges to a point $r_t \in X$. As the metric is continuous, we must have that

$$d(p, r_t) = Ct$$

so if the map $\gamma(t) = r_t$ is a curve, we get the desired map. But this map is continuous, essentially by definition! Indeed, any function on a dense subset of $[0,1]$ has a unique continuous extension given exactly as we have described it.

ii) For any partition

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$$

we have, by construction, that

$$d(\gamma(t_{i+1}), \gamma(t_i)) = (t_{i+1} - t_i)C$$

and so

$$\sum_{i=1}^{n} d(\gamma(t_{i+1}), \gamma(t_i)) = C$$

for all partitions. As such $L_d(\gamma) = C$.

iii) We know that $d_L(p,q) \geq d(p,q)$, so the claim follows immediately as we can find a curve whose length is $d(p,q)$, and so $d(p,q) \geq d_L(p,q)$.

iv) Yes! The exact same argument works, though we have to be a bit careful with our "in-between" points. For any points p, q, we can find r such that

$$d(p,r) + d(r,q) = d(p,q), \quad \text{and so} d(p,r) = Ct_r$$

for some $0 < t_r < 1$. By taking repeated in-between points, we obtain a set of "in-between" values in $[0,1]$. For any $t \in [0,1]$, we can find a sequence of in-between values converging to it as follows:

- Let $t_{r_{-1}} = 0$ and $t_{r_0} = 1$
- Assume we have found t_{r_n}. Then, t is between t_{r_n} and one of $t_{r_{n-1}}$ or $t_{r_{n-2}}$ (Call it $t_{r_{n-i}}$
- Then we find an in-between point r_{n+1} between r_n and r_{n-i}, and hence $t_{r_{n+1}}$.
- Iterating this, we find a sequence $\{t_{r_n}\}$ which we claim converges to t.

Once we can show that this sequence converges, the exact same construction works. This sequence doesn't always converge, but it can be split into an increasing sequence that converges to some $L \leq t$ and a decreasing sequence that converges to some $R \geq t$. As X is complete, we get corresponding points in X, and can begin the process again, constructing a new (hopefully convergent) sequence of points in-between r_L and r_R. Iterating this gives an infinite sequence of intervals $[L_i, R_i]$ containing t, and the length of these intervals is decreasing. If the length of these intervals is bounded below by a positive number, we can take the limit of the L_i and the limit of the R_i and start over. This can be made formal using a transfinite induction argument, but eventually, we get a sequence of points in X converging to t.