MAU22302/33302 - Euclidean and non-Euclidean Geometry

Homework 3

Trinity College Dublin

Course homepage Answers are due for March 22$^{\text{th}}$, 23:59.

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Triangulations of polygons with holes (50 pts)*

1. (15pts) Show that in a polygon with a single hole, there exists a straight line between a vertex on the external boundary and a vertex on the internal boundary such that the interior of the line is contained entirely in the interior of the polygon;

   Hint: Think back to our proof of diagonals using leftmost-ness

2. (15pts) By "cutting" the polygon along such a line, we obtain a polygon without holes. Use this to show that a polygon with a single hole has a strong triangulation, and give an expression for the number of triangles in terms of the number of vertices.

3. (20pts) Show that such a line exists in a polygon with many holes. By induction on the number of holes, show that every polygon with holes admits a strong triangulation and give an expression for the number of triangles in terms of the number of vertices and holes.

Solution 1

1. We present two solutions. One is based on a sort of ear-clipping approach (and could be modified to work for diagonals), while the other relies only on an orientation of the plane.

   (a) Let P be a polygon with a single hole. We first consider the case where the outer boundary of P is a triangle. In this case, choose an orientation of the plane so that P has a leftmost outer vertex v. Let w be the leftmost inner vertex. As w is interior to the triangle, the line vw cannot cross the outer boundary. As w is leftmost, and v is necessarily to the left of w, vw cannot cross the inner boundary.

   Now suppose P has more than 3 outer vertices. Then the outer boundary has an ear. If the diagonal cutting off this ear either does not intersect the inner boundary, it divides P into a triangle without holes and a polygon P' with a hole and fewer vertices in its outer boundary. We can suppose P' has a line of the desired form by inductive reasoning.

   If the diagonal does intersect the inner boundary, then we orient the plane so that this diagonal is vertical, and the apex v of the ear is to the left. Then, let w be the leftmost vertex of the inner boundary. We must have w is to the left of the diagonal. As w is leftmost and v is further left, vw cannot intersect the inner boundary, and as w is contained within the ear, vw cannot intersect the outer boundary, giving a line as needed.

   (b) Let P be a polygon with a single hole. Pick an orientation of the plane and let v be a leftmost vertex of the inner boundary. (By rotating our orientation slightly, we could assume that v is the leftmost vertex of the inner boundary) There must be a vertex of the outer boundary that is strictly to the left of v. Otherwise, the edge joining the two leftmost vertices of the outer boundary would be to the right of v and hence would intersect an edge from v. Thus, there is at least one outer vertex strictly to the left of v. Let w be the rightmost such vertex. Reflecting the plane if necessary, we also assume that w is (non-strictly) above v.

   If w is the only vertex to the left of v, then vw must be contained entirely within P, as an edge of the exterior boundary intersecting

vw must have points to the left of v and hence one endpoint to the left of v.

If vw is contained entirely within P, then we are done. Otherwise, consider the exterior edge intersecting vw closest to v. This divides the plane into a w-side and a v-side. Let z be a rightmost vertex left of v in the v-side of the plane. Then if vz intersects the exterior boundary in an edge, this edge cannot intersect vw, as otherwise z would be in the w-side of the plane. Thus one endpoint of this edge is right of vz, but left of vw. Furthermore, this endpoint must be in the v-side of the plane, as otherwise it would intersect the external boundary. But this contradicts z being rightmost among such vertices. Thus vz is contained entirely within P.

2. We have a number of ways to handle "cutting" along this line that do not fundamentally change the geometry, that we will discuss in a moment. For purposes of this homework, you can freely assume that something like cutting out a small rectangle around the cut line, or pretending that the cut line exists as a double line is fine.

   By separating the cut line vw into two lines v_1w_1 and v_2w_2, we obtain a new polygon P' with $n+2$ vertices and no holes, as the two lines connect the inner and outer boundaries. Thus, we obtain a standard polygon, which can be strongly triangulated into $(n+2)-2 = n$ triangles.

   To make this cutting precise (which was not required), we have a couple of options. One is to extend the cut line to a point p_0 in the interior of the hole. Considering this p_0 to be the origin, we can then left our polygon to a polygon on the helicoid

$$\{(\rho\cos(\theta), \rho\sin(\theta), \theta) \in \mathbb{R}^3 \mid \rho \geq 0, \theta \in \mathbb{R}\}$$

   As we follow our polygon around the origin, the two pieces of the cut line will end up on different levels of the helicoid - so v_1w_1 will have $z = 0$ which v_2w_2 will have $z = 2\pi$. The helicoid is homeomorphic to the plane, and we can reproduce all results about triangulations for polygons on the helicoid. This gives one approach.

   Another approach is to consider the map in which we again fix p_0 as the origin and vw along the positive x-axis, and consider the map $\mathbb{R}^2 \to \mathbb{R}^2$

given in polar coordinates by

$$(r,\theta) \mapsto \left(r, \frac{\theta}{2}\right)$$

This map sends lines to lines, so our polygon is maps to a new polygon, with a missing edge v_2w_2 along the negative x-axis. Adding this edge in, we obtain a regular polygon that we can triangulate. Upon applying the inverse map, we obtain a triangulation of P, having sent v_2w_2 back to vw.

The final approach is probably the most intuitive. Take some small $\varepsilon > 0$ and take points w_1, w_2 on the edges incident to w at distance ε, and points v_1, v_2 on the edges incident to v at distance ε. Joining v_iw_i and deleting v, w, we obtain a new polygon without holes whose geometry is very close to that of P. We can triangulate this and consider the limit as $\varepsilon \to 0$. There is some risk here that in the limit, a diagonal will suddenly pass through a vertex it did not before, or that a triangle will collapse into a degenerate triangle, but it is not hard to should that the number of triangles will remain constant.

3. Again, there are a number of approaches, but we will only give one. To find such an edge, we choose an orientation of the plane so that there is a leftmost inner vertex w. As it is leftmost, any edges from w going left will not intersect any inner boundary. As such, we can repeat the arguments of part (1) to find the desired edge.

   We claim there exists a strong triangulation with $n - 2 + 2h$ triangles in a polygon with n vertices and h holes. We proceed by induction.

   Suppose the claim holds true for all polygons with $h - 1$ holes. Then let P be a polygon with n vertices and h holes. By finding a line from an interior vertex to an exterior vertex contained entirely within the interior of P, we can "cut" P along this line to obtain a polygon with $n + 2$ vertices and $h - 1$ holes. By induction, this has a strong triangulation with $n + 2(h - 1) = n - 2 + 2h$ triangules, which we can glue back together to get a strong triangulation of P.

Exercise 2 *Simple spherical triangles (50 pts)*

In the first two problems, you may find it helpful to think back to the Euclidean setting.

1. (20 pts) We call a proper spherical triangle ABC isosceles if $|AB| = |AC|$. Show that the spherical angles $\angle ABC$ and $\angle ACB$ are equal.

2. (15 pts) Given a spherical line segment AB with midpoint M, the perpendicular bisector of AB is the great circle through M whose defining plane is perpendicular to that of AB. Show that every point on the perpendicular bisector is equidistant from A and B.

3. (15 pts) Given two spherical line segments of equal length ℓ_1 and ℓ_2, show there exists an isometry f such that $f(\ell_1) = \ell_2$, i.e. the isometry group of the sphere acts transitively on the set of spherical line segments of fixed length.

   Hint: It may help to start by showing we can map the great circle C_1 to the great circle C_2. Isometries of the sphere are given by reflections and rotations!

Solution 2

1. As ABC is proper, we can extend the edges AB and AC through B and C to points edges AD and AE with D not interior to AB and E not interior to AC. We assume $|AE| \geq |AD|$ and choose a point F on BD. We can then find G on CE such that $|AF| = |AG|$, for example, by rotating the line segment AF around the spherical diameter through A. As SAS congruence holds in spherical geometry, we have

   $$\triangle AFC = \triangle AGB$$

   and so $|FC| = |GB|$, and

   $$\angle BFC = \angle AFC = \angle AGB = \angle CGB.$$

   We also have, by construction, that $|BF| = |CG|$. Thus, by SAS congruence, we have

   $$\triangle BFC = \triangle CGB$$

   and hence $\angle FBC = \angle GCB$. Thus

   $$\angle ABC = \angle ABF - \angle FBC - \angle ACG - \angle GCB = \angle ACB.$$

2. Suppose P is on the perpendicular bisector. Then we claim that

$$\triangle PMA = \triangle PMB$$

as PM is common, $|MA| = |MB|$ by definition, and we have

$$\angle PMA = \frac{\pi}{2} = \angle PMB$$

Thus, by SAS congruence, the triangles are equal. In particular $|PA| = |PB|$.

3. Suppose ℓ_1 is part of the great circle C_1 and ℓ_2 is part of the great circle C_2. These circles either coincide or intersect at exactly two antipodal points. If there intersect at exactly two antipodal points, we consider rotation around the spherical diameter ℓ connecting the points of intersection.

   The rotation of a great circle is a great circle, and any such rotation fixes the points of intersection. Picking a point P on C_2, there exists a rotation around ℓ sending P to a point on C_1. The image of this rotation is a circle with three points on C_1, and hence is equal to C_1.

   We have therefore reduced to the case of $C_1 = C_2$. Now consider rotations around the pole of C_1. Orient the plane defining C_1 so that each line segment $\ell_i = A_iB_i$ with B_i anticlockwise from A_i. Then, there is a rotation around the pole sending A_2 to A_1. As rotation preserves orientation, the image of ℓ_2 will lie along ℓ_1, and as they are the same length, the lines must coincide. Thus, the group of isometries acts transitively on the set of line segments of a given length.

   Alternatively, if we do not want to choose an orientation, we can pick a rotation taking an endpoint of ℓ_2 to an endpoint of ℓ_1 and then, if necessary, reflecting in a perpendicular plane to ensure that the image of ℓ_2 lies along ℓ_1.

Exercise 3 *Optional - Regular polyhedra and footballs (0 pts)*

Recall that we saw in Tutorial 6, for any triangulation $\mathcal{T}$ of the sphere, the relation

$$V(\mathcal{T}) - E(\mathcal{T}) + F(\mathcal{T}) = 2$$

held. We will use this to classify all regular polyhedra

1. Let $\mathcal{P}$ be a subdivision of the sphere into (proper) spherical polygons. Show that
$$V(\mathcal{P}) - E(\mathcal{P}) + F(\mathcal{P}) = 2$$
This claim holds for any subdivision, regardless of properness vs improperness of the polygons, but it may be easier to show for proper polygons.

2. A polyhedron is a 3-dimensional surface made of vertices, joined by edges that only intersect in vertices, which are arranged into the boundaries of of polygonal faces, such that the faces intersect only in edges, and at most two faces have an edge in common. They divide space into a bounded interior region and an unbounded exterior region.

   Or at least, that will do for us. As Grünbaum said:

   > The Original Sin in the theory of polyhedra goes back to Euclid, and through Kepler, Poinsot, Cauchy and many others ... at each stage ... the writers failed to define what are the polyhedra

   We call a polyhedron regular if each of its faces are congruent regular polygons, i.e. all sides and angles in the faces are equal.

   Show that a (regular) polyhedron is homeomorphic to a sphere, and give an a homeomorphism thus that the edges of the polyhedron induce a polygonal subdivision of the sphere.

3. Hence or otherwise, show there are exactly 5 regular polyhedra, given by the 5 platonic solids.

4. Hence or otherwise, show that any football stitched from pentagons and hexagons must use exactly 12 pentagons. This will hold regardless of how regular the polygons are!

Solution 3

1. Suppose P is a polygon in our subdivision and let C be a great circle intersecting the interior of P. Then C cuts P into two polygons P_1 and

P_2. Taking the new subdivision $\mathcal{P}'$ obtained by splitting P via C, we note that we have increased the face count by 1, and the edge count by 1. If the new edge splits any edges in two, we obtain a corresponding new vertex. Thus

$$[V(\mathcal{P}) - E(\mathcal{P}) + F(\mathcal{P}) = [V(\mathcal{P}') - E(\mathcal{P}') + F(\mathcal{P}')$$

Thus, we can freely subdivide our subdivision. For a non-proper faces of our subdivision, choose a great circle and cut the face into several proper faces according to the intersection of the face and the great circle. This gives a new subdivision with fewer non-proper faces. Repeating this we can reduce to the case of a subdivision into proper polygons. We can then triangulate all such proper polygons to find that

$$V(\mathcal{P}) - E(\mathcal{P}) + F(\mathcal{P}) = V(\mathcal{T}) - E(\mathcal{T}) + F(\mathcal{T}) =$$

for some triangulation into proper triangles $\mathcal{T}$. From tutorial 6, the RHS is equal to 2, so the claim follows.

2. Choose a point in the interior of the polyhedron K to be the origin, and define a map $\rho : K \to \mathbb{S}^2_1$ by sending x to the intersection of the ray $0 - x$ with the sphere. This is clearly continuous with continuous inverse, giving a homeomorphism.

   Note also that any edge of the polyhedron defines a plane through the origin, and that the image of this edge is the intersection of this plane with the sphere. Thus, the edges of the polyhedron are mapped to spherical lines. Thus, the faces of the polyhedron are mapped to spherical polygons and so the edges induce a polygonal subdivision of the sphere.

3. We therefore have that for any polyhedron K

$$V(K) - E(K) + F(K) = 2$$

   For a regular polyhedron, every face has the same number of edges and every edge appears in two faces, so

$$2E = nF$$

Similarly, every vertex is contained in the same number of faces, and every face contains the same number of vertices:

$$kV = nF$$

Thus

$$\left(\frac{n}{k} - \frac{n}{2} + 1\right) F = 2.$$

and it is sufficient to find integer solutions to this. Equivalently, we need to find integer solutions to

$$(2-k)nF + 2kF = 2k$$

such that $2|nF$ and $k|nF$.

We can slightly simplify our search by noting that, by regularity, our polyhedron must be convex and so the total sum of angles meeting at every vertex is at most 2π. As $k \geq 3$, and an n-gon has internal angle $\frac{n-2}{n}\pi$, we must have

$$\frac{k(n-2)}{n}\pi \leq 2\pi$$

and hence

$$kn - 2k \leq 2n \quad \Rightarrow \quad n \leq \frac{2k}{k-2} \leq 6$$

for all $k \geq 3$. So, we consider $n = 3, 4, 5, 6$.

Rearranging the inequality, we also have

$$3 \leq k \leq \frac{2n}{n-2}$$

Thus, if $n = 6$, $k = 3$; if $n = 5$, $k = 3$; if $n = 4$, $k \in \{3, 4\}$; if $n = 3$, $k \in \{3, 4, 5, 6\}$.

Testing each case explicitly, we find a contradiction except for

$$(n, k) \in \{(5,3), (4,3), (3,3), (3,4), (3,5)\}$$

which correspond to the Platonic solids: the dodecahedron, the cube, the tetrahedron, the octahedron, and the icosahedron.

4. Oddly, the football is easier to consider. Suppose we have H hexagonal faces and P pentagonal faces. We must have

$$2E = 6H + 5P$$

as each edge appears in exactly two faces. For reasons of convexity, we must have that exactly three faces meet at each vertex, so

$$3V = 6H + 5P.$$

Hence

$$\frac{6H + 5P}{3} - \frac{6H + 5P}{2} + H + P = 2$$

which simplifies to

$$\frac{P}{6} = 2$$

and hence $P = 12$.

If we include no hexagons, we get a dodecahedron. A standard football will include 20 hexagons. If you include too many hexagons, you get something less ball shaped and more polygon. Specifically, you'll approximate what is called the polar dual of a dodecahedron, also known as an icosahedron!