MAU22302/33302 - Euclidean and non-Euclidean Geometry

Homework 2

Trinity College Dublin

Course homepage Answers are due for March 4^{th}, 23:59.

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Affine transformations and geometric objects (50 pts)*

1. (20pts) Determine which, if any, of the following is an affine transformation (not necessarily an isometry)

   (a) $(x, y) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$,

   (b) $(x, y) \mapsto \left(\frac{ax+b}{c}, \frac{cy+d}{a}\right)$,

   (c) $(x, y) \mapsto (-x, y)$ if $x \geq 0$ and $(x, y) \mapsto (x, -y)$ if $x < 0$.

2. (20pts) We will call a class of objects "geometric" if the image of a object under an isometry of $\mathbb{R}^n$ is in the same class. Recall we defined $A \subset \mathbb{R}^n$ to be an affine space of dimension $k \leq n$ if there is a vector subspace $V \subset \mathbb{R}^n$ of dimension k and a vector $a \in \mathbb{R}^n$ such that

$$A = \{a + v \mid v \in V\}$$

Show that the class of affine spaces of dimension k is geometric.

Hint: Don't forget to check that the dimension doesn't change. PME students only need to consider lines and planes in $\mathbb{R}^3$, unless they choose to do so otherwise.

3. (10pts) Show that the class of ellipses

$$\left\{E = \{(x,y) \mid Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0\} \mid B^2 - 4AC < 0\right\}$$

is geometric.

Hint: Apply a generic isometry $(x,y) \mapsto M(x,y) + v$. What does $M^T M = I$ imply about relations satisfied by its entries?

Solution 1

1. (a) This is not affine. It is not defined at $(0,0)$, and does not preserve lines: take for example the line $y = 1$, which gets sent to

$$\left\{\left(\frac{x}{\sqrt{x^2+1}}, \frac{1}{\sqrt{x^2+1}}\right) \mid x \in \mathbb{R}\right\}$$

which is decidedly not line-like.

(b) This is affine! It is the map $v \mapsto Av + w$ where

$$A = \begin{pmatrix} \frac{a}{c} & 0 \\ 0 & \frac{c}{a} \end{pmatrix}, \quad w = \begin{pmatrix} \frac{b}{c} \\ \frac{d}{a} \end{pmatrix}$$

(c) This is not affine. It's not even continuous. We could consider the line $y = 1$ again which maps to the disjoint union

$$\{(x,1) \mid x \leq 0\} \sqcup \{x, -1 \mid x < 0\}.$$

2. Let A be an affine space and F and isometry. We know $Fx = Mx + b$ for some orthogonal matrix M and vector b. Then, as

$$FA = \{Ma + Mv + b \mid v \in V\} = \{q + w \mid w \in MV\}$$

letting $q = Ma + b$ and $w = Mv$. Then as M is an invertible linear map, MV is a vector subspace of dimension k, so we get exactly an affine space as needed. Hence the class of affine spaces of dimension k is geometric.

3. We consider the generic isometry $(x,y) \mapsto \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)(x,y)+v$. As the matrix is orthogonal, we have that

$$a^2 + b^2 = 1,\ ac + bd = 0,\ c^2 + d^2 = 1,\ ad - bc = \pm 1$$

Letting $v = (p, q)$, and letting (z, w) be the image of (x, y) under our isometry, we find

$$x = a(z - p) + c(w - q),\ y = b(z - p) + d(w - q)$$

Filling this into

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

and expanding, we find

$$(Aa^2+Bab+Cb^2)z^2+(2Aac+Bad+Bbc+2Cbd)zw+(Ac^2+Bcd+Cd^2)w^2+D'z+E'w+F' = 0$$

for some constants D', E', F'. So for this to be an ellipse, we just need to check that

$$(2Aac + Bad + Bbc + 2Cbd)^2 < 4(Aa^2 + Bab + Cb^2)(Ac^2 + Bcd + Cd^2)$$

Expanding this out, we want that

$$4A^2a^2c^2+4C^2b^2d^2+B^2(a^2d^2+b^2c^2+2abcd)+8AC(abcd)+4AB(a^2cd+abc^2)+4BC(abd^2+b^2cd)$$

is less than

$$4A^2a^2c^2+4C^2b^2d^2+4B^2(abcd)+4AC(a^2d^2+b^2c^2)+4AB(a^2cd+abc^2)+4BC(abd^2+b^2cd)$$

Rearranging this inequality, we find we want

$$B^2(a^2d^2 - 2abcd + b^2c^2) < 4AC(a^2d^2 - 2abcd + b^2c^2)$$

which is

$$B^2(ad - bc)^2 < 4AC(ad - bc)^2$$

which is true as long as $ad - bc \neq 0$. So actually, ellipse are preserved under affine transformations too! (Though the eccentricity will change under generic affine transformations)

We could also show that if the images of a set of points form an ellipse, then the original points form an ellipse. This way we wouldn't have to invert the relationship. We could also translate to the origin and consider a generic reflection. That would also suffice

Exercise 2 *Reflecting on reflections (50 pts)*

In general, the order in which we apply isometries matters: $f \circ g \neq g \circ f$. If $f \circ g = g \circ f$, then we say f and g commute. Let's explore what kinds of isometries commute, focusing on reflections.

1. (10 pts) Either by explicit computation, or carefully drawing diagram, give an example of a pair of reflections in intersecting lines that do not commute.

2. (10 pts) Either by explicit computation, or carefully drawing diagram, give an example of a pair of reflections in intersecting lines that do commute.

3. (15 pts) Show that reflections in two parallel lines does not commute.

   Hint: Pick a nice point and keep track of what side of the lines it ends up on

4. (15 pts) Show that if ℓ_1 and ℓ_2 are distinct intersecting lines (you may assume the point of intersection is the origin), then

   $$r_{\ell_1} \circ r_{\ell_2} = r_{\ell_2} \circ r_{\ell_1}$$

   if and only if ℓ_1 and ℓ_2 are perpendicular. You may use either analytic or synthetic approaches.

Solution 2

1. Consider reflection r_1 in the x-axis and r_2 in the line $x = y$. It is not hard to see that

   $$r_1(x, y) = (x, -y),\ r_2(x, y) = (y, x)$$

   We just consider their action on the point $(1, 1)$. Reflection in $x = y$ fixes $(1, 1)$, so $r_1 \circ r_2$ maps $(1, 1)$ to $(1, -1)$. We then have

   $$r_2(r_1(1, 1)) = (-1, 1) \neq (1, -1)$$

   This can also be achieved by drawing a moderately accurate picture, as even a quite sketchy drawing will reveal that the two images will be in different quadrants.

2. The simplest example is given by the reflections in the coordinate axes

$$r_1(x, y) = (-x, y),\ r_2(x, y) = (x, -y)$$

which clearly commute.

3. Without loss of generality, we can assume the parallel lines are $y = 0$ and $y = c$ for some $c > 0$. We achieve this by applying an isometry. (We could technically assume $c = 1$, but that would require a similarity transformation)

   Our argument essentially relies on the fact that reflection in $y = 0$ swaps positive and negative y-coordinates, while reflection in $y = c$ swaps points with $y < c$ and points with $y > c$.

   Take a point with $-c < y < 0$. Reflecting first in $y = 0$ sends it to a point with $0 < y < c$, and then reflecting in $y = c$ sends it to a point with $y > c$.

   Reflecting first in $y = c$ sends it to a point with $y > c$, and then reflecting in $y = 0$ sends it to a point with $y < -c < 0$. Hence, reflection in parallel lines does not commute.

   Here, again, a sufficiently accurate picture would suffice!

4. We can assume, without loss of generality, that one of the lines is the x-axis, and the other is a line at an angle of θ from the x-axis. We can furthermore assume that they intersect in the origin.

   Reflection in the x-axis is given by

$$r_1(x, y) = (x, -y)$$

   while reflection in a line at angle θ is given by

$$r_2(x, y) = (\cos(2\theta)x + \sin(2\theta)y, \sin(2\theta)x - \cos(2\theta y)).$$

   Thus

$$\begin{aligned} r_1(r_2(x, y)) &= (\cos(2\theta)x + \sin(2\theta)y, -\sin(2\theta)x + \cos(2\theta y)) \\ r_2(r_1(x, y)) &= (\cos(2\theta)x - \sin(2\theta)y, \sin(2\theta)x + \cos(2\theta y)) \end{aligned}$$

These are equal for all x, y, if and only if $\sin(2\theta) = 0$, which occurs if and only if $2\theta = k\pi$ for some integer k. Thus, the reflections commute if and only if

$$\theta = \frac{k\pi}{2}$$

for some integer k. If k is even, the lines coincide, but we have assumed otherwise. If k is odd, the lines are perpendicular, as required.

For a more synthetic approach, I will only provide a sketch, leaving out some "obvious" geometry. Let P be a point, and let P_1 be the intersection of $Pr_1(P)$ with ℓ_1, P_2 be the intersection of $Pr_2(P)$ with ℓ_2, P_3 be the intersection of $P_1r_2(P_1)$ with ℓ_2, and P_4 be the intersection of $P_2r_1(P_2)$ with ℓ_1. Let Q be the intersection of ℓ_1 and ℓ_2. Then, it is not hard to show that if ℓ_1 and ℓ_2 are perpendicular

$$PP_1QP_2,\ r_1(P)P_1Qr_1(P_2)\,r_2(P)P_2Qr_2(P_1)$$

are all congruent figures, as are the images under $r_2 \circ r_1$ and $r_1 \circ r_2$:

$$r_2(r_1(P))r_2(P_1)Qr_1(P_2),\ r_1(r_2(P)r_1(P_2)Qr_2(P_1)$$

Two congruent quadrilaterals which coincide on two adjacent edges must coincide everywhere (or by following a series of congruent triangles involving a diagonal), we see that the points

$$r_2(r_1(P)),\ r_1(r_2(P))$$

must coincide.

Conversely, if these points coincide, then we have a quadrilateral

$$Pr_1(P)r_1(r_2(P))r_2(P)$$

which must have opposite sides being equal lengths, and is therefore a parallelogram. However, if we extend the lines $Pr_1(P)$ and $r_2(P)r_1(r_2(P))$ to intersect, we obtain what must be an isosceles triangle. Hence our parallelogram has equal angles along a single side. It therefore has all angles equal and is a rectangle.

The lines ℓ_1 and ℓ_2 join the midpoints of the sides of this rectangle and must therefore be perpendicular.

Exercise 3 *Optional - Coxeter groups (0 pts)*

A Coxeter group is a group generated by a (usually finite) set of involutions $\{r_1, r_2, \ldots\}$, i.e. elements such that

$$r_i^2 = \mathrm{id},$$

called reflections, such that for $i \neq j$, either

$$(r_i r_j)^{m_{ij}} = \mathrm{id}$$

for some integer $m_{ij} \geq 2$, or

$$(r_i r_j)^n \neq \mathrm{id}$$

for any $n \geq 1$ (we often say $m_{ij} = \infty$ in this case).

Show that $\mathrm{Isom}(\mathbb{R}^2)$ is a Coxeter group with an infinite set of generators, and determine m_{ij} for every pair of reflections r_i, r_j.

Hint: For a given pair of intersecting lines, you can choose an affine coordinate system such that the associated reflections are given by matrices

Solution 3

We take as our infinite set of generators the set of all reflections (returning later to why this must be infinite). As such, our indices should be thought of as elements of the uncountable set of lines.

If r_i and r_j correspond to a pair of intersecting lines, then there is a coordinate system, and angles θ_i, θ_j such that

$$r_i r_j \leftrightarrow \begin{pmatrix} \cos(2\theta_i) & \sin(2\theta_i) \\ \sin(2\theta_i) & -\cos(2\theta_i) \end{pmatrix} \begin{pmatrix} \cos(2\theta_j) & \sin(2\theta_j) \\ \sin(2\theta_j) & -\cos(2\theta_j) \end{pmatrix}$$

Multiplying the matrices and using some trigonometric identities, we find $r_i r_j$ corresponds to a rotation by an angle of

$$2\theta_{ij} := 2(\theta_i - \theta_j)$$

around the point of intersection. Then $(r_i r_j)^n$ is a rotation by an angle of $2n\theta_{ij}$. This is equal to the identity if

$$2n\theta_{ij} = 2\pi m$$

for some integer $m \in \mathbb{Z}$. Thus, m_{ij} is finite if and only if

$$\theta_{ij} = \frac{m\pi}{n}$$

for some integers m, n. If such integers exist, m_{ij} is given by the minimal such n.

If r_i and r_j correspond to a pair of parallel lines (which we assume do not coincide), then there is a coordinate system in which

$$r_i(x, y) = (x, -y),\ r_j(x, y) = (x, 2c - y)$$

for some constant c. Then

$$r_i r_j(x, y) = (x, y - 2c)$$

and so

$$(r_i r_j)^n(x, y) = (x, y - 2cn).$$

Clearly, this can never be the identity for $n \geq 1$, and so $m_{ij} = \infty$ for parallel lines.

Finally to see that we need infinitely many generators, the simplest argument is a cardinality argument. We know that translation is an isometry, and that there are uncountably many translations. A finite set of generators can generate a group with at most countably many elements. Indeed, we can assign a "size" to any element of a Coxeter group given by the length of a minimal factorisation into generators, and there can be at most g^l elements of length l with a generating set of size g. Thus, we require infinitely many generators to produce our uncountably many isometries.