MAU22302/33302 - Euclidean and non-Euclidean Geometry

Homework 1

Trinity College Dublin

Course homepage Answers are due for February 11^{th}, 23:59.

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Playfair's Postulate (50pts)*

Euclid's formulation of the fifth postulate is rather lengthy:

> *If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines - if produced indefinitely - meet on that side on which are the angles less than two right angles.*

It usually replaced by Playfair's postulate:

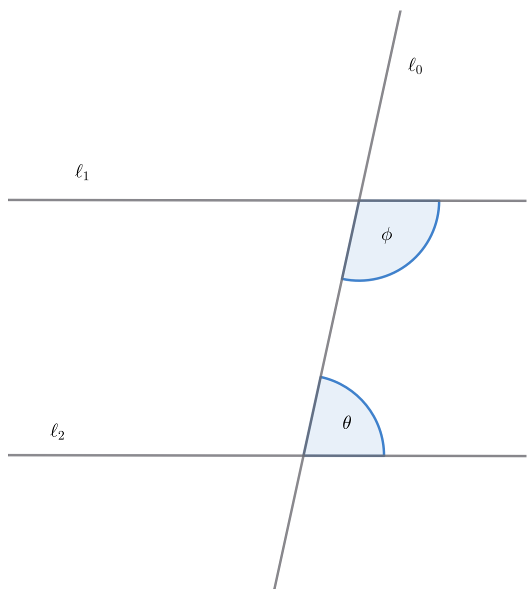
> *Given a line ℓ and a point P not on ℓ, there exists exactly one line through P that does not intersect ℓ, if both lines are extended indefinitely, i.e. there is a unique line parallel to ℓ through P.*

The goal of this question is to establish their equivalence. You may freely use any of the postulates and and any of propositions 1-27 given at the end of the assignment. In particular, you may assume at least one line parallel to ℓ through a point P exists.

1. (20pts) Assume Postulates 1-5, and consider a line ℓ and a point P not on ℓ. By considering two lines through P, show that at least one intersects ℓ. Conclude Playfair's Postulate.

   Hint: Drop a perpendicular from P

2. (20pts) Assume Postulates 1-4 and Playfair's Postulate. Establish that, if ℓ_1 and ℓ_2 are parallel, then $\phi + \theta$ is equal to two right angles in the diagram below.



   Hint: Suppose otherwise and construct another line such that the corresponding angles add to two right angles

3. (10pts) Hence conclude Postulate 5.

Solution 1

1. Suppose we have two distinct lines ℓ_1 and ℓ_2 through P. Draw a perpendicular to ℓ from P and consider the angles θ_i between the perpendicular and ℓ_i on a given side of the perpendicular. As ℓ_1 and ℓ_2 are distinct, $\theta_1 \neq \theta_2$ and in particular, at least one of them is not a right angle. Without loss of generality, we take it to be θ_1.

Furthermore, we can assume θ_1 is less than a right angle. Otherwise, we consider the angles on the other side of the perpendicular. Hence the sum of angles on that side of the perpendicular is less than two right angles, and so by Postulate 5, ℓ_1 and ℓ must intersect. In general, given two lines through P, at least one must intersect ℓ, and so we can have at most one parallel line. Proposition 27 implies at least one parallel line exists, and hence exactly one parallel line exists.

2. Let $\ell_1 \cap \ell = P$ and $\ell_2 \cap \ell = Q$. By Proposition 23, we can construct an line through P at an angle equal to θ, so that the interior angle is $\pi - \theta$. If this line is not equal to ℓ_1, then it cannot be parallel to ℓ_2 and hence must intersect ℓ_2 at a point R. Then the triangle PQR contains two angles θ and $\pi - \theta$ (regardless of which side of ℓ R is on) adding to two right angles, contradicting Proposition 17. Hence, we must have that this line is ℓ_2 and hence $\pi - \theta = \phi$ as needed.

3. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines cannot be parallel, as the interior angles for parallel lines add to two right angles. Hence they intersect, if extended indefinitely, giving Postulate 5.

Exercise 2 *Triangles and Parallelograms (50pts)*

In the following you may freely assume any of the given postulates and propositions, as well as your preferred parallel postulate from the first exercise

1. (10pts) Establish that the sum of angles in a triangle is equal to two right angles

   Hint: Introduce some alternate angles

2. (10pts) Let ℓ_1 and ℓ_2 be parallel lines, with points A, B on ℓ_1 and C, D on ℓ_2, appearing in that order. Suppose $|AB| = |CD|$. Show that AC is parallel to BD.

   Hint: BC27

3. (15pts) Let ℓ_1 and ℓ_2 be parallel lines. Let A, B, E, F be points on ℓ_1, appearing in that order. Let C, D be points on ℓ_2 in that order.

Suppose

$$|AB| = |CD| = |EF|$$

Show that the parallelograms $ABDC$ and $EFDC$ have equal areas.

Hint: Write each parallelogram as a sum/difference of triangles

4. (15pts) Let $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ be a norm satisfying parallelogram rule

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2$$

Show that the function

$$I(x,y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$

is additive in the first variable, i.e.

$$I(x+y,z) = I(x,z) + I(y,z)$$

for all $x, y, z \in \mathbb{R}^n$.

Not-a-hint: Draw a picture in $\mathbb{R}^2$ to figure out why parallelogram rule is called that. It won't help with the problem, but it's nice to know!

Hint: You'll need to combine four instances of the parallelogram rule

5. (0pts) Show that if $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$f(x+y) = f(x) + f(y)$$

then $f(qx) = qf(x)$ for all $q \in \mathbb{Q}$.

Suppose further that f is continuous. Show $f(\lambda x) = \lambda f(x)$ for all $\lambda \in \mathbb{R}$. Hence conclude the function defined in the previous part is linear in the first variable.

Solution 2

1. Let ABC be the triangle. Using Proposition 31, we can construct a line parallel to BC through A. Let P be a point on this line on the same side of A as B, and Q a point on the same side as C. By Proposition 29, we have that

$$\angle CBA = \angle BAP, \quad \text{and} \quad \angle BCA = \angle CAQ$$

Clearly

$$\angle BAP + \angle BAC + \angle CAQ = \angle PAQ = \pi$$

from which the claim follows.

2. Draw the line BC. By Proposition 29,

$$\angle ABC = \angle DCB$$

As $|AB| = |CD|$ and $|BC| = |BC|$, we conclude that $ABC = DCB$ as triangles. Hence

$$\angle ACB = \angle DBC$$

and so by Proposition 27, AC is parallel to BD.

3. Let X be the intersection of CE and BD. Then

$$|ABDC| = |AEC| + |CXD| - |BEX|$$

and

$$|EFDC| = |BFD| + |CXD| - |BEX|$$

so it is sufficient to show $|AEC| = |BFD|$. We know

$$|AE| = |AB| + |BE| = |EF| + |BE| = |BF|$$

From the arguments of the previous part, we also have

$$|AC| = |BD| \quad \text{and} \quad |EC| = |FD|$$

Hence, by Proposition 8, AEC and BFD are equal as triangles and the claim follows.

4. We have that

$$\begin{aligned}
\|x+y+z\|^2 &= -\|x+z-y\|^2 + 2\|x+z\|^2 + 2\|y\|^2,\\
\|x+y-z\|^2 &= -\|x-z-y\|^2 + 2\|x-z\|^2 + 2\|y\|^2,\\
\|x+y+z\|^2 &= -\|y+z-x\|^2 + 2\|y+z\|^2 + 2\|x\|^2,\\
\|x+y-z\|^2 &= -\|y-z-x\|^2 + 2\|y-z\|^2 + 2\|x\|^2.
\end{aligned}$$

Adding the first and third, and subtracting the second and fourth, we find

$$2\|x+y+z\|^2 - 2\|x+y-z\|^2 = 2\|x+z\|^2 - 2\|x-z\|^2 + 2\|y+z\|^2 - 2\|y-z\|^2$$

and so

$$I(x+y,z) = I(x,z) + I(y,z)$$

5. We must have $f(0) = 0$, taking $x = y = 0$. It's an easy induction to see that $f(nx) = nf(x)$ for $n \in \mathbb{Z}$ $n \geq 0$, and then as $f(x) + f(-x) = f(0) = 0$, we extend this to all $n \in \mathbb{Z}$. As

$$f(x) = f\left(n \cdot \frac{x}{n}\right) \quad \Rightarrow f(x) = nf\left(\frac{x}{n}\right)$$

we find $f(x/n) = f(x)/n$ for all $n \in \mathbb{Z}$, $n \neq 0$. Combining these quickly gives the result for rational multiples.

If f is continuous at a point λx, then taking a sequence r_n of rational numbers converging to λ (which exists as $\mathbb{Q}$ is dense in $\mathbb{R}$), we find that

$$f(\lambda x) = \lim_{n\to\infty} f(r_n x) = \lim_{n\to\infty} r_n f(x) = \lambda f(x).$$

Fixing $z \in \mathbb{R}^n$, the function $f(x) = I(x, z)$ is additive and continuous, and is hence linear.

Postulates and propositions - slightly rephrased

Postulates

1. There exists a line between any two points
2. A finite line segment can be extended indefinitely
3. Given a point and a line segment from that point, a circle with centre the given point and radius the given line segment can be drawn
4. All right angles are equal

Propositions

1. One can construct an equilateral triangle on any base.
2. One can construct a line of length equal to a given line, from a given point
3. One can construct a line of length equal to the difference of two given line segments
4. If two triangles have two sides equal to two sides, and the angles contained are equal, then the triangles are equal
5. In an isosceles triangle, the angles at the base are equal

8. If two triangles have three sides equal to three sides, the triangles are equal
9. One can bisect a given angle
10. One can bisect a given finite line
11. One can draw a straight line at right angles to a given straight line from a given point on the line
12. One can draw a straight line at right angles to a given (infinite) straight line from a given point not on it.
13. If a straight line intersects a straight line, then it makes angles whose sum equals two right angles on the same side of a given line.

16. In angle triangle, if one of the sides is extended, the exterior angle is greater than either of the interior opposite angles
17. In any triangle, the sum of any two angles is less than two right angles.
23. One can construct an angle equal to a given angle on a given straight line and at a point on it.
27. If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another
29. A straight line falling on two parallel straight lines makes the alternate angles equal to one another and the exterior angle equal to the interior opposite angle
31. Given a straight line and a point not on the line, one can draw a parallel line through the point