MAU22302/33302 - Euclidean and non-Euclidean Geometry

Homework 4

Trinity College Dublin

Course homepage Answers are due for April 8^{th}, 23:59.

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Möbius transformations*

i) Determine the image of $z = 1+2i$ (in the form $a+bi$) under the Möbius transformations given by the matrices

$$\gamma_1 = \begin{pmatrix} 4 & 3 \\ 0 & 2 \end{pmatrix}, \ \gamma_2 = \begin{pmatrix} 0 & 1 \\ -2 & 8 \end{pmatrix}, \ \gamma_3 = \begin{pmatrix} 1 & 1 \\ 2 & 7 \end{pmatrix}.$$

ii) Determine a real matrix γ with determinant 1 such that the corresponding Möbius transformation satisfies

$$\gamma \cdot i = \frac{8+i}{5}, \ \gamma \cdot (2i) = \frac{7+i}{4},$$

Hint: Reduce to linear equations in the matrix coefficients and rescale

iii) Show that if $\gamma, \delta \in \mathrm{SL}_2(\mathbb{R})$, then $\gamma \cdot (\delta \cdot z) = (\gamma\delta) \cdot z$, so that composition of Möbius transformations is compatible with matrix multiplication.

iv) Show that $\gamma \cdot z \in \mathbb{H}^2$ if $z \in \mathbb{H}^2$, for every $\gamma \in \mathrm{SL}_2(\mathbb{R})$, i.e. the action of the special linear group preserves the upper half plane.

Exercise 2 *Computing integrals trigonometrically*

Let $P = (\cos(b), \sin(b))$ and $Q = (\cos(a), \sin(a)$ be points on the semicircle σ of radius 1 centred at the origin, with $0 < a < \frac{\pi}{2} < b < \pi$, and let $\ell = (0, y)$ be the y-axis.

i) Determine a circle C centred on the x-axis such that inversion in C maps P and Q to points on ℓ

   Hint: When does inversion send circles to line?

ii) Using Euclidean trigonometry, determine the images $\iota_C(P)$ and $\iota_C(Q)$.

   Hint: What lines must the images lie on? How could we find the angles those lines make with the x-axis?

iii) Hence determine $d_{\mathbb{H}}(P, Q)$ and conclude the identity

$$\tanh^{-1}(\cos(a)) - \tanh^{-1}(\cos(b)) = \ln\left(\frac{\tan\left(\frac{b}{2}\right)}{\tan\left(\frac{a}{2}\right)}\right)$$

   Hint: You may freely use that $\int \frac{dx}{\sin(x)} = -\tanh^{-1}(\cos(x)) +$ constant. This problem also gives an incredibly roundabout way of proving a half angle formula!

Exercise 3 *Optional: Hyperbolic curvature*

Recall that for a geometry in which a circle with centre p and radius s has circumference $\mathrm{Circ}(C_p(s))$, we defined the Gaussian curvature at a point to be

$$K(p) = \lim_{s\to 0^+} 3\frac{2\pi s - \mathrm{Circ}(C_p(s))}{\pi s^3}$$

Let's determine the curvature of $\mathbb{H}^2$.

i) Let ABC be a hyperbolic triangle with $|AB| = |AC| = s$ (hyperbolically) and hyperbolic angle $\angle ABC = \alpha$, determine the hyperbolic length of BC.

   Hint: Think back to tutorial 8!

ii) By approximating the circle via a hyperbolic n-gon, hence compute $\mathrm{Circ}(C)$ as a function of s (and possibly (A, B)).

   Hint: What is $\lim_{n\to\infty} \sinh(an)/n$*?*

iii) Using whatever calculus is necessary, determine $K((A, B))$ for $\mathbb{H}^2$

   Having hopefully found that $\mathbb{H}^2$ has fundamentally different curvature to Euclidean space, lets consider whether we can still approximate hyperbolic circles well by Euclidean circles for very small or large radii

iv) Determine $\lim \frac{\mathrm{Circ}(C)}{2\pi s}$ as s tends to 0 or ∞.

v) If either limit tends to 1, this suggests the hyperbolic circle with small/large radius is very close to a Euclidean circle with the same centre and radius. But a hyperbolic circle *is* a Euclidean circle, with a different centre and radius. In the appropriate limiting case, do these centres and radii get close?

vi) If feeling incredibly brave, explicitly parametrise the hyperbolic circle and determine its circumference via our integral expression for hyperbolic length. The integral you get can either be evaluated using the Residue Theorem, or using the substitution $u = \tan\left(\frac{x}{2}\right)$ and some care around discontinuities

Exercise 4 *Optional - Lengths in convex metric spaces*

For (X, d) a metric space, we can assign a length to any sufficiently nice curve $\gamma : [0, 1] \to X$ as follows

$$L_d(\gamma) := \sup_P \sum_{i=1}^{n} d(\gamma(t_{i+1}), \gamma(t_i))$$

where we take the supremum over partition of $[0, 1]$. This length could be infinite, but modulo this, serves as a good length function for sufficiently nice γ. From this, we can define a new metric

$$d_L(p, q) = \inf_{\gamma:p\to q} L_d(\gamma)$$

where we take the infimum over sufficiently nice paths from p to q. Using the triangle inequality, it is not hard to show

$$d_L(p, q) \geq d(p, q)$$

but when is it equal? We claim this holds if X is complete and convex.

We say a metric space X is convex if, for every $p, q \in X$ there exists $r \in X$ such that

$$d(p, q) = d(p, r) + d(r, q)$$

We will call X a midpoint metric space if such an r exists with $d(p, r) = d(r, q)$.

i) Let $C = d(p, q)$. Show that if X is a complete midpoint metric space, there exists a curve $\gamma : [0, 1] \to X$ such that

$$\gamma(0) = p, \quad \gamma(1) = q$$
$$d(p, \gamma(t)) + d(\gamma(t), q) = d(p, q) \text{ for all } 0 \leq t \leq 1$$
$$d(p, \gamma(t)) = Ct.$$

ii) Show that such a curve γ has $L_d(\gamma) = C$.

iii) Hence conclude that $d_L = d$ in a complete midpoint metric space.

iv) Does the same hold in a complete convex metric space? (For sake of your sanity, do not try to be rigorous about this one)