MAU22302/33302 - Euclidean and non-Euclidean Geometry

Homework 3

Trinity College Dublin

Course homepage Answers are due for March 22^{th}, 23:59.

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Triangulations of polygons with holes (50 pts)*

1. (15pts) Show that in a polygon with a single hole, there exists a straight line between a vertex on the external boundary and a vertex on the internal boundary such that the interior of the line is contained entirely in the interior of the polygon;

   Hint: Think back to our proof of diagonals using leftmost-ness

2. (15pts) By "cutting" the polygon along such a line, we obtain a polygon without holes. Use this to show that a polygon with a single hole has a strong triangulation, and give an expression for the number of triangles in terms of the number of vertices.

3. (20pts) Show that such a line exists in a polygon with many holes. By induction on the number of holes, show that every polygon with holes admits a strong triangulation and give an expression for the number of triangles in terms of the number of vertices and holes.

Exercise 2 *Simple spherical triangles (50 pts)*

In the first two problems, you may find it helpful to think back to the Euclidean setting.

1. (20 pts) We call a proper spherical triangle ABC isosceles if $|AB| = |AC|$. Show that the spherical angles $\angle ABC$ and $\angle ACB$ are equal.

2. (15 pts) Given a spherical line segment AB with midpoint M, the perpendicular bisector of AB is the great circle through M wholes defining plane is perpendicular to that of AB. Show that every point on the perpendicular bisector is equidistant from A and B.

3. (15 pts) Given two spherical line segments of equal length ℓ_1 and ℓ_2, show there exists an isometry f such that $f(\ell_1) = \ell_2$, i.e. the isometry group of the sphere acts transitively on the set of spherical line segments of fixed length.

   Hint: It may help to start by showing we can map the great circle C_1 to the great circle C_2. Isometries of the sphere are given by reflections and rotations!

Exercise 3 *Optional - Regular polyhedra and footballs (0 pts)*

Recall that we saw in Tutorial 6, for any triangulation $\mathcal{T}$ of the sphere, the relation

$$V(\mathcal{T}) - E(\mathcal{T}) + F(\mathcal{T}) = 2$$

held. We will use this to classify all regular polyhedra

1. Let $\mathcal{P}$ be a subdivision of the sphere into (proper) spherical polygons. Show that

   $$V(\mathcal{P}) - E(\mathcal{P}) + F(\mathcal{P}) = 2$$

   This claim holds for any subdivision, regardless of properness vs improperness of the polygons, but it may be easier to show for proper polygons.

2. A polyhedron is a 3-dimensional surface made of vertices, joined by edges that only intersect in vertices, which are arranged into the boundaries of of polygonal faces, such that the faces intersect only in edges,

and at most two faces have an edge in common. They divide space into a bounded interior region and an unbounded exterior region.

Or at least, that will do for us. As Grünbaum said:

> The Original Sin in the theory of polyhedra goes back to Euclid, and through Kepler, Poinsot, Cauchy and many others ... at each stage ... the writers failed to define what are the polyhedra

We call a polyhedron regular if each of its faces are congruent regular polygons, i.e. all sides and angles in the faces are equal.

Show that a (regular) polyhedron is homeomorphic to a sphere, and give an a homeomorphism thus that the edges of the polyhedron induce a polygonal subdivision of the sphere.

3. Hence or otherwise, show there are exactly 5 regular polyhedra, given by the 5 platonic solids.

4. Hence or otherwise, show that any football stitched from pentagons and hexagons must use exactly 12 pentagons. This will hold regardless of how regular the polygons are!