MAU22302/33302 - Euclidean and non-Euclidean Geometry

Homework 2

Trinity College Dublin

Course homepage Answers are due for March 4^{th}, 23:59.

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Affine transformations and geometric objects (50 pts)*

1. (20pts) Determine which, if any, of the following is an affine transformation (not necessarily an isometry)

   (a) $(x, y) \mapsto \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right)$,

   (b) $(x, y) \mapsto \left(\frac{ax+b}{c}, \frac{cy+d}{a} \right)$,

   (c) $(x, y) \mapsto (-x, y)$ if $x \geq 0$ and $(x, y) \mapsto (x, -y)$ if $x < 0$.

2. (20pts) We will call a class of objects "geometric" if the image of a object under an isometry of $\mathbb{R}^n$ is in the same class. Recall we defined $A \subset \mathbb{R}^n$ to be an affine space of dimension $k \leq n$ if there is a vector subspace $V \subset \mathbb{R}^n$ of dimension k and a vector $a \in \mathbb{R}^n$ such that

$$A = \{a + v \mid v \in V\}$$

Show that the class of affine spaces of dimension k is geometric.

Hint: Don't forget to check that the dimension doesn't change. PME students only need to consider lines and planes in $\mathbb{R}^3$, unless they choose to do so otherwise.

3. (10pts) Show that the class of ellipses

$$\left\{E = \{(x,y) \mid Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0\} \mid B^2 - 4AC < 0\right\}$$

is geometric.

Hint: Apply a generic isometry $(x,y) \mapsto M(x,y) + v$. What does $M^T M = I$ imply about relations satisfied by its entries?

Exercise 2 *Reflecting on reflections (50 pts)*

Reflections which commute Reflections which don't commute Coxeter Criterion?

In general, the order in which we apply isometries matters: $f \circ g \neq g \circ f$. If $f \circ g = g \circ f$, then we say f and g commute. Let's explore what kinds of isometries commute, focusing on reflections.

1. (10 pts) Either by explicit computation, or carefully drawing diagram, give an example of a pair of reflections in intersecting lines that do not commute.

2. (10 pts) Either by explicit computation, or carefully drawing diagram, give an example of a pair of reflections in intersecting lines that do commute.

3. (15 pts) Show that reflections in two parallel lines does not commute.

   Hint: Pick a nice point and keep track of what side of the lines it ends up on

4. (15 pts) Show that if ℓ_1 and ℓ_2 are distinct intersecting lines (you may assume the point of intersection is the origin), then

$$r_{\ell_1} \circ r_{\ell_2} = r_{\ell_2} \circ r_{\ell_1}$$

   if and only if ℓ_1 and ℓ_2 are perpendicular. You may use either analytic or synthetic approaches.

Exercise 3 *Optional - Coxeter groups (0 pts)*

A Coxeter group is a group generated by a (usually finite) set of involutions $\{r_1, r_2, \ldots\}$, i.e. elements such that

$$r_i^2 = \mathrm{id},$$

called reflections, such that for $i \neq j$, either

$$(r_i r_j)^{m_{ij}} = \mathrm{id}$$

for some integer $m_{ij} \geq 2$, or

$$(r_i r_j)^n \neq \mathrm{id}$$

for any $n \geq 1$ (we often say $m_{ij} = \infty$ in this case).

Show that $\mathrm{Isom}(\mathbb{R}^2)$ is a Coxeter group with an infinite set of generators, and determine m_{ij} for every pair of reflections r_i, r_j.

Hint: For a given pair of intersecting lines, you can choose an affine coordinate system such that the associated reflections are given by matrices