MAU22203/33203 - Analysis in Several Real Variables

Tutorial Sheet 4

Trinity College Dublin

Course homepage

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Computing derivatives*

i) Using limits, verify that the derivative of $f(x, y) = x^2 - 18xy + y^2$ is equal to the Jacobian at all points (p, q).

ii) Define a map $\varphi : \mathbb{R}^8 \to \mathbb{R}^4$ via matrix multiplication: if

$$\varphi(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), f_3(\vec{x}), f_4(\vec{x}))$$

the components are determined by

$$\begin{pmatrix} f_1(\vec{x}) & f_3(\vec{x}) \\ f_2(\vec{2}) & f_4(\vec{4}) \end{pmatrix} = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} \begin{pmatrix} x_5 & x_7 \\ x_6 & x_8 \end{pmatrix}.$$

Determine $(\mathrm{D}\,\varphi)_{\vec{x}}$.

iii) Hence show that, viewing (2×2)-matrices as elements of $\mathbb{R}^4$,

$$(\mathrm{D}(AB))_t = (\mathrm{D}\,A)_t B(t) + A(t)(\mathrm{D}\,B)_t$$

for any differentiable function $A, B : \mathbb{R} \to \mathbb{R}^4$.

Solution 1

i) The Jacobian at (p, q) is given by

$$(\mathrm{J}\, f)_{(p,q)} = (2p - 18q,\, 2q - 18p).$$

We therefore want to show that

$$\lim_{(x,y)\to(p,q)} \frac{f(x, y) - f(p, q) - (\mathrm{J}\, f)_{(p,q)}((x - p, y - q))}{\|(x - p, y - q)\|} = 0$$

or equivalently

$$\lim_{(h,k)\to(0,0)} \frac{f(p + h, q + k) - f(p, q) - (\mathrm{J}\, f)_{(p,q)}(h, k)}{\|(h, k)\|} = 0.$$

The numerator is given by

$$2ph+h^2+2qk+k^2-18pk-18qh-18hk-(2p-18q)h-(2q-18p)k = h^2+k^2-18hk$$

and so we want to show that

$$\lim_{(h,k)\to(0,0)} \frac{h^2 + k^2}{\sqrt{h^2 + k^2}} - 18\frac{hk}{\sqrt{h^2 + k^2}} = 0.$$

The first term is equal to $\sqrt{h^2 + k^2}$, which tends to 0, so it suffices to show that

$$-18 \lim_{(h,k)\to(0,0)} \frac{hk}{\sqrt{h^2 + k^2}} = 0$$

or, equivalently

$$\lim_{(h,k)\to(0,0)} \frac{2|hk|}{\sqrt{h^2 + k^2}} = 0.$$

To see this, not that

$$(|h| - |k|)^2 \geq 0 \quad \Leftarrow \quad h^2 + k^2 \geq 2|hk|.$$

Thus, we have that

$$0 \leq \frac{2|hk|}{\sqrt{h^2 + k^2}} \leq \frac{h^2 + k^2}{\sqrt{h^2 + k^2}} = \sqrt{h^2 + k^2}$$

and so, by the squeeze theorem, the limit as $(h, k) \to (0, 0)$ is equal to 0, as needed.

ii) We will compute the Jacobian matrix of

$$\varphi(\vec{x}) = \begin{pmatrix} x_1x_5 + x_3x_6 & x_1x_7 + x_3x_8 \\ x_2x_5 + x_4x_6 & x_2x_7 + x_4x_8 \end{pmatrix}$$

which turns out to be

$$(\mathrm{J}\,\varphi)_{\vec{x}} = \begin{pmatrix} x_5 & 0 & x_6 & 0 & x_1 & x_3 & 0 & 0 \\ 0 & x_5 & 0 & x_6 & x_2 & x_4 & 0 & 0 \\ x_7 & 0 & x_8 & 0 & 0 & 0 & x_1 & x_3 \\ 0 & x_7 & 0 & x_8 & 0 & 0 & x_2 & x_4 \end{pmatrix}.$$

This has continuous components everywhere. Thus, φ is differentiable everywhere with derivative equal to the Jacobian.

iii) By chain rule, the derivative of the function $A(t)B(t) = \varphi(A(t), B(t))$ is given by

$$(\mathrm{J}\,\varphi)_{(A(t),B(t)} \begin{pmatrix} (\mathrm{D}\,A)_t \\ (\mathrm{D}\,B)_t \end{pmatrix}$$

The derivatives of A and B are given by

$$(\mathrm{D}\,A)_t = \begin{pmatrix} A_1'(t) \\ A_2'(t) \\ A_3'(t) \\ A_4'(t) \end{pmatrix} \quad (\mathrm{D}\,B)_t = \begin{pmatrix} B_1'(t) \\ B_2'(t) \\ B_3'(t) \\ B_4'(t) \end{pmatrix}$$

and hence $(\mathrm{D}\,AB)_t$ is the vector

$$\begin{pmatrix} B_1A_1' + B_2A_3' + A_1B_1' + A_3B_2' \\ B_1A_2' + B_2A_4' + A_2B_1' + A_4B_2' \\ B_3A_1' + B_4A_3' + A_1B_3' + A_3B_4' \\ B_3A_2' + B_4A_4' + A_2B_3' + A_4B_4' \end{pmatrix}$$

which we can rewrite as the matrix

$$\begin{pmatrix} B_1A_1' + B_2A_3' + A_1B_1' + A_3B_2' & B_3A_1' + B_4A_3' + A_1B_3' + A_3B_4' \\ B_1A_2' + B_2A_4' + A_2B_1' + A_4B_2' & B_3A_2' + B_4A_4' + A_2B_3' + A_4B_4' \end{pmatrix}$$

which is equal to

$$\begin{pmatrix} A_1' & A_3' \\ A_2' & A_4' \end{pmatrix} \begin{pmatrix} B_1 & B_3 \\ B_2 & B_4 \end{pmatrix} + \begin{pmatrix} A_1 & A_3 \\ A_2 & A_4 \end{pmatrix} \begin{pmatrix} B_1' & B_3' \\ B_2' & B_4' \end{pmatrix}$$

Exercise 2 *Inverse Function Theorem*

i) Call a continuously differentiable function $f : \mathbb{R}^m \to \mathbb{R}^m$ locally invertible at $\vec{p}$ if there exists an open set $U \subset \mathbb{R}^m$ containing $\vec{p}$ such that $f : U \to f(U)$ is a bijection with a continuously differentiable inverse. Is

$$f(x, y) = \begin{pmatrix} e^{xy} \\ \sin(y + x) \end{pmatrix}$$

locally invertible at $(0, \pi)$?

ii) Let A be the set of $(x, y) \in \mathbb{R}^2$ such that f is locally invertible at (x, y). Is A open, closed, or neither in $\mathbb{R}^2$?

Solution 2

i) We first compute the derivative of f via the Jacobian

$$(\mathrm{D}\, f)_{(x,y)} = \begin{pmatrix} ye^{xy} & xe^{xy} \\ \cos(x + y) & \cos(x + y) \end{pmatrix}$$

This has continuous components, so f is continuously differentiable. At $(0, \pi)$, this is equal to

$$\begin{pmatrix} \pi & 0 \\ -1 & -1 \end{pmatrix}$$

which has determinant $-\pi \neq 0$. Hence, by the inverse function theorem, f is locally invertible at $(0, \pi)$.

ii) If f is locally invertible at (x, y), then there exists continuously differentiable $\mu : f(U) \to U$ such that $\mu(f(x, y))$ for all $(x, y) \in f(U)$. Chain rule tells us that

$$(\mathrm{D}\, \mu)_{f(x,y)}(\mathrm{D}\, f)_{(x,y)} = I$$

and so $(\mathrm{D}\, f)_{(x,y)}$ is invertible. The inverse function theorem tells us that if $(\mathrm{D}\, f)_{(x,y)}$ is invertible, then f is locally invertible. Hence A is the set

$$\{(x, y) \mid \det(\mathrm{D}\, f)_{(x,y)} \neq 0\} = \{(x, y) \mid (y - x)e^{xy} \cos(x + y) \neq 0\}$$

which is open.

Exercise 3 *The implicit function theorem*

i) Let
$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$
Show that, for every $(x_0, y_0) \in S^1$, there exists open $V \subset \mathbb{R}^2$ such that $V \cap S^1$ is homeomorphic to an open $U \subset \mathbb{R}$.

ii) Show that there exists open $V \subset \mathbb{R}^2$ such that $V \cap S^1$ is homeomorphic to an open interval $(a, b) \subset \mathbb{R}$.

Hint: Think about connectedness

iii) Let
$$A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2,\ x + y + z = 0\}.$$
Show that, for all but finitely many $(x, y, z) \in A$, there exists open $V \subset \mathbb{R}^3$ such that $V \cap A$ is homeomorphic to an open $U \subset \mathbb{R}$.

iv) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. Suppose that for each $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $f(x, y) = 0$. Denote this y by $c(x)$. Suppose further that $\partial_y f(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$. Show that c is differentiable, with derivative
$$c'(x) = -\frac{\partial_x f(x, c(x))}{\partial_y f(x, c(x)}.$$

Solution 3

i) By the implicit function theorem, it suffices to show that the derivative of $f(x, y) = x^2 + y^2 - 1$ has rank 1 everywhere.
$$(\mathrm{D}\, f)_{(x,y)} = (2x\ 2y)$$
which clearly has rank at most 1, and can only have rank 0 if $x = y = 0$, which is not a point on S^1. Thus, the claim holds by the implicit function theorem.

ii) The homeomorphism $\psi : V \cap S \to U$ induced by the implicit function theorem will induce a homeomorphism $\psi : V' \cap S \to \psi(V' \cap S)$ for any open $V' \subset V$. In particular, we can restrict ψ to the connected component of V containing (x, y). Since connectedness is preserved by homeomorphisms, the image must be an open connected set in $\mathbb{R}$, i.e. an open interval.

iii) The functions

$$f_1(x,y,z) = x^2 + y^2 - z^2,$$
$$f_2(x,y,z) = x + y + z$$

are continuously differentiable, so we can apply the implicit function theorem: it suffices to determine where the matrix

$$\mathrm{J}(x,y,z) = \begin{pmatrix} 2x & 2y & -2z \\ 1 & 1 & 1 \end{pmatrix}$$

has rank 2. It cannot have rank 0, and if it has rank 1, then each of the columns must be a scalar multiple of a common vector $\vec{v}$. Since the second component is 1 in each case, we must have $2x = 2y = -2z$, and so $J(x,y,z)$ has rank 2 unless $(x,y,z) = (t,t,-t)$ for some $t \in \mathbb{R}$. If $(x,y,z) \in A$ and $(x,y,z) = (t,t,-t)$, then

$$t + t - t = 0 \quad \Leftarrow \quad t = 0.$$

Thus, away from $(0,0,0)$, the implicit function theorem applies and the claim follows.

iv) The conditions given in the question imply that the implicit function theorem applies, with there existing open $V_{x_0} \subset \mathbb{R}^2$ containing

$$(x_0, c(x_0)) \in \{(x,y) \mid f(x,y) = 0\} = S$$

open $U \subset \mathbb{R}$, and continuously differentiable $g_{x_0} : U \to \mathbb{R}$ such that

$$V_{x_0} \cap S = \{(x, g_{x_0}(x)) \mid x \in U\}.$$

The uniqueness of c implies that $c(x) = g_{x_0}(x)$ for every $x \in U$, and that c is differentiable at x_0. Since this is true for every $x_0 \in \mathbb{R}$, we have that c is differentiable.

Since $f(x, c(x)) = 0$ for every $x \in \mathbb{R}$, we can apply chain rule to show that

$$\partial_x f(x, c(x)) + \partial_y f(x, c(x)) c'(x) = 0$$

from which the claim follows.

Exercise 4 *The Challenge*

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with $|f'(x)| \le c < 1$ for all $x \in \mathbb{R}$. Define $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\phi(x, y) = (x + f(y), y + f(x)).$$

1. Show that ϕ is continuously differentiable

2. Show the conditions of the inverse function theorem apply, and hence ϕ has a local inverse $\mu_{(p,q)}$ near every point $(p, q) \in \phi(\mathbb{R}^2)$

3. Show that ϕ is injective and therefore there exists $\mu : \phi(\mathbb{R}^2) \to \mathbb{R}^2$ such that $\phi(\mu(p, q)) = (p, q)$.

   Hint: Suppose otherwise and find a way to use the mean value theorem

4. Show that $\phi(\mathbb{R}^2)$ is open and explain why μ must be continuously differentiable.

5. Show that ϕ is surjective.

   Hint: can you rewrite surjectivity as a fixed point condition, or show the image is closed?

Solution 4

1. The Jacobian of ϕ is
$$\begin{pmatrix} 1 & f'(y) \\ f'(x) & 1 \end{pmatrix}$$
   which has continuous entries, and so ϕ is continuously differentiable.

2. The determinant of the derivative of ϕ is $1 - f'(x)f'(y) > 0$ as from the bounds on the derivative. Hence $(\mathrm{D}\,\phi)_{(x,y)}$ is invertible everywhere, and the inverse function theorem applies. Thus ϕ has a local inverse in some open around every point $(p, q) \in \mathbb{R}^2$.

3. Note that, by the mean value theorem
$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\xi) \le c < 1$$

and so $f(x_2) - f(x_1) < x_2 - x_1$ for all $x_1 < x_2$. As such

$$|f(x) - f(p)| < |x - p|$$

for all $x \neq p \in \mathbb{R}$. Suppose $\phi(x, y) = \phi(p, q)$ with $x \neq p$, $y \neq q$. Then we have that

$$0 = x - p + f(y) - f(q) \quad \text{and} \quad y - q + f(x) - f(p) = 0$$

and so

$$|x - p| = |f(y) - f(q)| < |y - q| \quad \text{and} \quad |y - q| = |f(x) - f(p)| < |x - p|$$

which is impossible, as it implies $|x - p| < |x - p|$. Thus, we must have $x = p$ or $y = q$. If either of these hold, then the equalities also imply the other, so we must have $(x, y) = (p, q)$. Hence ϕ is injective.

Therefore, there exists $\mu : \phi(\mathbb{R}^2) \to \mathbb{R}^2$ such that $\phi(\mu(p, q)) = (p, q)$.

4. By the inverse function theorem, around every $(p, q) \in \phi(\mathbb{R}^2)$, there exists an open set $U_{(p,q)}$ and a continuously differentiable function $\mu_{(p,q)} : U_{(p,q)} \to \mathbb{R}^2$ such that $\phi(\mu_{(p,q)}(x, y)) = (x, y)$. Hence $U_{(p,q)} \subset \phi(\mathbb{R}^2)$, and we can write

$$\phi(\mathbb{R}^2) \subset \bigcap_{(p,q)\in\phi(\mathbb{R}^2)} U_{(p,q)} \subset \phi(\mathbb{R}^2).$$

Therefore the image is equal to this union and hence is open.

Furthermore, by the injectivity of ϕ, there can by at most one function $\eta : U_{(p,q)} \to \mathbb{R}^2$ such that $\phi(\eta(x, y)) = (x, y)$. As both $\mu_{(p,q)}$ and $\mu|_{U_{(p,q)}}$ satisfy this condition, they must be equal. Thus $\mu|_{U_{(p,q)}}$ is continuous differentiable. As the $U_{(p,q)}$ cover $\phi(\mathbb{R}^2)$, and continuous differentiability is a local condition, this implies μ is continuously differentiable everywhere.

5. We give two arguments. For the first, fix $(p, q) \in \mathbb{R}^2$ and note that $(p, q) \in \phi(\mathbb{R}^2)$ is equivalent to the function

$$\Phi(x, y) = (p - f(y), q - f(x))$$

having a fixed point somewhere in $\mathbb{R}^2$. As $\mathbb{R}^2$ is closed, it is enough to show Φ is a contraction:

$$\begin{aligned}\|\Phi(x,y)-\Phi(a,b)\| &= \|(f(b)-f(y),f(a)-f(x))\| \\ &= \sqrt{(f(b)-f(y)^2+(f(a)-f(x))^2} \\ &\leq \sqrt{c^2(y-b)^2+c^2(a-x)^2} = c\|(x,y)-(a,b)\|.\end{aligned}$$

Thus a fixed point exists, and so ϕ is surjective. Hence, ϕ is a bijective continuously differentiable map with continuously differentiable inverse (a diffeomorphism!) from $\mathbb{R}^2$ to itself.

For the second argument, we want to show $\phi(\mathbb{R}^2)$ is closed. Let $\vec{z}$ be a limit point of $\phi(\mathbb{R}^2)$. There is a sequence $\{\vec{z}_n\} \subset \phi(\mathbb{R}^2)$ such that $\lim_{n\to\infty} \vec{z}_n = \vec{z}$. Letting $\vec{x}_n = \mu(\vec{z}_n)$ we obtain a sequence in $\mathbb{R}^2$ which must converge to a point $\vec{x} \in \mathbb{R}^2$ by continuity of μ. Then by continuity of ϕ, $\vec{z} = \phi(\vec{x})$, so $\vec{z} \in \phi(\mathbb{R}^2)$. Hence $\phi(\mathbb{R}^2)$ contains all its limits points and is therefore closed. But we have already showed that it is open. The only non-empty subset of $\mathbb{R}^2$ that is both open and closed is $\mathbb{R}^2$ itself, so ϕ must be surjective.