MAU22203/33203 - Analysis in Several Real Variables

Tutorial Sheet 2

Trinity College Dublin

Course homepage

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Convergent subsequences*

Let $\{\vec{x}_k\}$ be a bounded sequence in $\mathbb{R}^m$ that does not converge to $\vec{0}$.

i) Show that there exist $R, c > 0$ such that $\{\vec{x}_k\}$ contains a subsequence contained within the set

$$\{\vec{x} \in \mathbb{R}^m \mid c \leq \|\vec{x}\| \leq R\},$$

ii) Show that this subsequence contains a convergent subsequence, converging to a point not equal to $\vec{0}$,

iii) Hence conclude that $\{\vec{x}_k\}$ contains a convergent subsequences, converging to a point other than $\vec{0}$.

Solution 1

i) Since $\{\vec{x}_k\}$ is bounded, by definition there exists R such that $\|\vec{x}_k\| \leq R$ for every $k \geq 1$. As $\{\vec{x}_k\}$ does not converge to $\vec{0}$, there exists some $c > 0$ such that for every $N > 0$, there exists $j_N \geq N$ such that $\|\vec{x}_{j_N}\| \geq c$. The subsequence given by these points $\{\vec{x}_{j_N}\}$ is contained within the given set.

ii) The sequence $\{\vec{x}_{j_N}\}$ is a bounded sequence, and so by the Bolzano-Weierstrass theorem, contains a convergent subsequence $\{\vec{x}_{j_{N_k}}\}$. This subsequence is contained entirely in the set

$$\{\vec{x} \in \mathbb{R}^m \mid c \leq \|\vec{x}\| \leq R\}$$

which is closed as it is the preimage of the closed interval $[c, R]$ under the continuous map $\vec{x} \mapsto \|\vec{x}\|$. The limit of a convergent sequence contained in a closed set is also contained within the closed set, so

$$\lim_{k\to\infty} \vec{x}_{j_{N_k}} \in \{\vec{x} \in \mathbb{R}^m \mid c \leq \|\vec{x}\| \leq R\}$$

and hence cannot be $\vec{0}$.

iii) The sequence $\{\vec{x}_{j_{N_k}}\}$ is a subsequence of $\{\vec{x}_k\}$ converging to a point other than $\vec{0}$.

Exercise 2 *Interiors*

Given a subset $X \subset \mathbb{R}^m$, we define its interior X^o, closure $\overline{X}$, and boundary ∂X by

$$X^o = \bigcup_{\substack{U \subset X \\ U \text{ open}}} U,$$

$$\overline{X} = \bigcap_{\substack{F \supset X \\ F \text{ closed}}} F,$$

$$\partial X = \overline{X} \setminus X^o.$$

i) Show that X^o is open and $\overline{X}$ is closed,

ii) Find the interior, closure, and boundary of

$$\{\vec{x} \in \mathbb{R}^m \mid \|\vec{x}\| \leq 2\},$$

iii) Find the interior, closure, and boundary of

$$\{(x, y) \in \mathbb{R}^2 \mid x \geq 2, y > 0\},$$

iv) Find the interior, closure, and boundary of

$$\{x \in \mathbb{R} \mid x \in \mathbb{Q}\}.$$

Hint: It might be easier to consider the interior of the complement.

Solution 2

i) X^o is a union of open sets and is therefore open. $\overline{X}$ is an intersection of closed sets and is therefore closed.

ii) Denote the set by X. Then, as $X = \overline{B}(\vec{0}, 2)$ is the closed ball of radius 2 centred at the origin, X is closed and so

$$\overline{X} = X \cap \bigcap_{F \supset X,\, F \neq X} F = X.$$

Clearly $X^o \supset B(\vec{0}, 2)$ contains the open ball. Now suppose it is not equal to it. Then there is a point $\vec{p} \in X$, which is contained within an open set contained within X, but $\|\vec{p}\| = 2$. This cannot occur, as then we would have an open ball of radius $\varepsilon > 0$ for some $\varepsilon > 0$, centred at $\vec{p}$ contained within X. But every such open ball contains a point $\vec{q} = \left(1 + \frac{\varepsilon}{2}\right)\vec{p}$ of norm $\|\vec{q}\| = 2 + \varepsilon$, not in X. Thus $X^o = B(\vec{0}, 2)$. Thus

$$\partial X = \{\vec{x} \in \mathbb{R}^m \mid \|\vec{x}\| = 2\}.$$

iii) The set

$$\{(x, y) \in \mathbb{R}^2 \mid x > 2, y > 0\} = \{(x, y) \in \mathbb{R}^2 \mid x > 2\} \cap \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$$

is open, as it is the intersection of two opens, and is contained within

$$X = \{(x, y) \in \mathbb{R}^2 \mid x \geq 2, y > 0\}.$$

If X^o contains any other points $(2, y)$, then it must contain an open ball around such a point. But any open ball around such a point will contain $(2 - \frac{\varepsilon}{2}, y)$, which is not in X, let alone X^o. Thus

$$X^o = \{(x, y) \in \mathbb{R}^2 \mid x > 2, y > 0\}.$$

To determine the closure, note that

$$\{(x, y) \in \mathbb{R}^2 \mid x \geq 2, y \geq 0\}$$

is closed, as it is the intersection of two closed sets. Furthermore, since X contains $(x, \frac{1}{n})$ for any $x \geq 2$, $n \geq 1$, so must any closed F containing X. The sequence $\{(x, \frac{1}{n})\}$ converges to $(x, 0)$, which must be contained in any closed F containing X. Thus, any closed F containing X also contains the closed set

$$\{(x, y) \in \mathbb{R}^2 \mid x \geq 2, y \geq 0\}$$

Thus this is the intersection of all closed sets containing X:

$$\overline{X} = \{(x, y) \in \mathbb{R}^2 \mid x \geq 2, y \geq 0\}$$

Finally note that

$$\partial X = \{(2, y) \in \mathbb{R}^2 \mid y \geq 0\} \cup \{(x, 0) \in \mathbb{R}^2 \mid x \geq 2\}.$$

iv) Note that, for any non-empty open set contained within $X = \mathbb{Q}$, there must be a point q of $\mathbb{Q}$ contained in this open set, and hence an open ball around q contained entirely within X. But any open ball around a rational number contains infinitely many irrational numbers, which are not in X. Thus the open subset of X are the empty set: $X^o = \emptyset$.

I claim the closure, and hence the boundary, are both equal to $\mathbb{R}$. To see this, note that

$$\mathbb{R} \setminus \overline{X} = (\mathbb{R} \setminus X)^o$$

and so it suffices to show that this interior is empty. Suppose $r \in \mathbb{R} \setminus \mathbb{Q}$. Then any open ball around r contains a rational approximation to r, and hence cannot be contained entirely within $\mathbb{R} \setminus \mathbb{Q}$. Thus $(\mathbb{R} \setminus \mathbb{Q})^o = \emptyset$.

Exercise 3 *Topological stuff*

i) Identify whether the following sets are open/closed in $\mathbb{R}^2$:

   a) $\{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$,
   b) $\{(x, y) \in \mathbb{R}^2 \mid x \neq 0, y \neq 0\}$,
   c) $\{(x, y) \in \mathbb{R}^2 \mid \max\{|x|, |y|\} = 1\}$,
   d) $\{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 17xy < 34\}$

ii) Suppose that $X \subset \mathbb{R}^m$ is not closed. Show that there exists a point $\vec{v} \in \mathbb{R}^m \setminus X$ such that

$$B(\vec{v}, \varepsilon) \cap X \neq \emptyset$$

for all $\varepsilon > 0$.

iii) Call a subset A of $\mathbb{Z}$ open if $\mathbb{Z} \setminus A$ is finite or if A is empty. Show that this collection of open sets defines a topology on $\mathbb{Z}$.

Solution 3

i) a) This is closed, as it is the preimage of the closed set $\{1\}$ under the continuous map $f(x, y) = xy$.

   b) This is open, as it us the intersection of the two open sets

$$\{(x, y) \in \mathbb{R}^2 \mid x \neq 0\}, \quad \text{and} \quad \{(x, y) \in \mathbb{R}^2 \mid y \neq 0\}$$

   which are open as they are the preimages of the open set $\mathbb{R}^{\setminus}\{0\}$ under the continuous maps $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ respectively.

   c) , This is closed as it is the union of four closed sets

$$\{(-1, y) \mid -1 \leq y \leq 1\} \quad \{\{(1, y) \mid -1 \leq y \leq 1\},$$
$$\{(x, -1) \mid -1 \leq x \leq 1\} \quad \{(x, 1) \mid -1 \leq x \leq 1\}.$$

   To see that these are closed, we will illustrate this with just one set

$$\{(-1, y) \mid -1 \leq y \leq 1\}$$

   which is the intersection of closed sets

$$\pi_1^{-1}(\{-1\}) \cap \pi_2^{-1}([-1, 1]).$$

d) This is open, as it is the intersection of the open sets

$$\{(x,y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2\}, \quad \{(x,y) \in \mathbb{R}^2 \mid 17xy < 34\},$$
$$\{(x,y) \in \mathbb{R}^2 \mid x^2 - 17xy + y^2 < 0\}$$

which are all preimages of open sets under continuous functions.

ii) Since X is not closed, $\mathbb{R}^m \setminus X$ is not open. This implies that there exists a point $\vec{v} \in \mathbb{R}^m \setminus X$ such that for every $\varepsilon > 0$, the open ball is not contained within $\mathbb{R}^m \setminus X$:

$$B(\vec{v}, \varepsilon) \not\subset \mathbb{R}^m \setminus X$$

which is exactly to say that for every $\varepsilon > 0$,

$$B(\vec{v}, \varepsilon) \cap X \neq \emptyset.$$

iii) The empty set is open, by definition. $\mathbb{Z}$ is open as $\mathbb{Z} \setminus \mathbb{Z} = \emptyset$ is finite. Now suppose we have a collection of open sets $\{A_i\}_{i \in I}$, which we can assume, without loss of generality, to be non-empty. Then $\mathbb{Z} \setminus A_i$ is finite and hence

$$\bigcap_{i \in I} \mathbb{Z} \setminus A_i = \mathbb{Z} \setminus \bigcup_{i \in I} A_i$$

must also be finite. Therefore

$$\bigcup_{i \in I} A_i$$

is open. Finally, suppose we have a finite collection of open sets $A_1, \ldots, A_k$. Then

$$\mathbb{Z} \setminus \bigcap_{i=1}^{k} A_i = \bigcup_{i=1}^{k} \mathbb{Z} \setminus A_i$$

is a finite union of finite sets, and is therefore finite. Hence

$$\bigcap_{i=1}^{k} A_i$$

is open, and we have a topology.

Exercise 4 *Continuous functions and convergent sequences*

i) Without resorting to an $\varepsilon - \delta$ proof, show that if $\phi_1, \phi_2 : \mathbb{R}^m \to \mathbb{R}^n$ are continuous functions, then

$$\vec{x} \mapsto \phi_1(\vec{x}) + \phi_2(\vec{x})$$

is a continuous function.

ii) Without resorting to an $\varepsilon - N$ argument, prove that if we have two sequences $\{\vec{x}_k\}$ and $\{\vec{y}_k\}$ in $\mathbb{R}^m$ converging to $\vec{p}$ and $\vec{q}$ respectively, then the sequence in $\mathbb{R}$ given by

$$\{z_k = \langle \vec{x}_k, \vec{y}_k \rangle\}$$

converges to $\langle \vec{p}, \vec{q} \rangle$.

iii) Let $X \subset \mathbb{R}^m$ be a strict subset of $\mathbb{R}^m$, and suppose we have a point $\vec{p} \in \mathbb{R}^m \setminus X$. Show that

$$\phi_{\vec{p}}(\vec{x}) = \frac{1}{\|\vec{x} - \vec{p}\|}$$

is continuous on X.

iv) Hence argue that if X is not closed in $\mathbb{R}^m$, there exists $\vec{p} \in \mathbb{R}^{\setminus} X$ such that the image $\phi_{\vec{p}}(X)$ is not bounded in $\mathbb{R}$.

Hint: Use 3.ii to construct a bad sequence.

Solution 4

i) The map

$$\psi(\vec{x}) = \phi_1(\vec{x}) + \phi_2(\vec{x})$$

is continuous if and only if all of its components are. The components of ψ are given by the sum of the components of ϕ_1 and ϕ_2, which are continuous. Thus the components of ψ are continuous, as the sum of continuous functions to $\mathbb{R}$ is continuous, and so ψ is continuous.

ii) Let $\vec{x}_{k,i}$ and $\vec{y}_{k,i}$ be the i$^{\text{th}}$ components of $\vec{x}_k$ and $\vec{y}_k$ respectively. The consider the sequence of points in $\mathbb{R}^{2m}$ given by

$$\{\vec{z}_k = (x_{k,1}, x_{k,2}, \ldots, x_{k,m}, y_{k,1}, \ldots, y_{k,m})\}.$$

Since it has convergent components, this sequences converges and it must converge to

$$(p_1, p_2, \ldots, p_m, q_1, \ldots, q_m).$$

Next note that the map $\phi : \mathbb{R}^{2m} \to \mathbb{R}$ given by

$$\phi(w_1, \ldots, w_m, w_{m+1}, \ldots, w_{2m}) = \sum_{i=1}^{m} w_i w_{m+1}$$

is continuous, and furthermore that

$$\phi(\vec{z}_k) = \langle \vec{x}_k, \vec{y}_k \rangle = z_k.$$

Thus, as ϕ is continuous,

$$\lim_{k\to\infty} z_k = \lim_{k\to\infty} \phi(\vec{z}_k) = \phi\left(\lim_{k\to\infty} \vec{z}_k\right) = \phi(p_1, \ldots, p_m, q_1, \ldots, q_m) = \langle \vec{p}, \vec{q} \rangle.$$

iii) We know that the map $\vec{x} \mapsto \vec{x} - \vec{p}$ is continuous, by part (i) of the question or by considering components. We know that the map $\vec{x} \mapsto \|\vec{x}\|$ is continuous, and so

$$\vec{x} \mapsto \|\vec{x} - \vec{p}\|$$

is continuous, as it is the composition of continuous functions. Finally note that this is non-zero for all $\vec{x} \in X$, and so we can conclude that $\phi_{\vec{p}}(\vec{x})$ is continuous on X.

iv) If X is not closed, then there exists a point $\vec{p} \in \mathbb{R}^m \setminus X$ such that $B(\vec{p}, \varepsilon) \cap X \neq \emptyset$ for every $\varepsilon > 0$. In particular, for every $k \geq 1$, there exists $\vec{x}_k \in X$ such that $\|\vec{x}_k - \vec{p}\| < \frac{1}{k}$. But then the sequence

$$\{\phi_{\vec{p}}(\vec{x}_k)\}$$

is unbounded, as $\phi_{\vec{p}}(\vec{x}_k) > k$, and is a subset of the image of $\phi_{\vec{p}}$, as needed.

Exercise 5 *A challenge from Spivak*

The following is a neat problem, where drawing pictures should make the answer obvious, though writing a good argument for it could be hard. Define a set $A \subset \mathbb{R}^2$ by

$$A = \{(x, y) \in \mathbb{R}^2 \mid 0 < x,\, 0 < y < x^2\}$$

i) Show that every straight line in $\mathbb{R}^2$ passing through $(0, 0)$ contains an open interval I such that $I \cap A = \emptyset$.

ii) Define

$$f(x, y) = \begin{cases} 1 \text{ if } (x, y) \in A, \\ 0 \text{ if } (x, y) \notin A. \end{cases}$$

Show that $f(x, y)$ is not continuous at $(0, 0)$.

iii) Fix a vector $(u, v) \in \mathbb{R}^2$ and define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) := f(tu, tv).$$

Show that g is continuous at 0.

iv) Show that, for any continuous $h : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that $h(0) = 0$, the function

$$f_h(x, y) = \begin{cases} 1 \text{ if } (x, y) \in A_h, \\ 0 \text{ if } (x, y) \notin A_h. \end{cases}$$

is not continuous at $(0, 0)$, while the function $g_h(t) = f_h(tu, tv)$ is continuous at 0. Here, we define

$$A_h := \{(x, y) \in \mathbb{R}^2 \mid 0 < x,\, 0 < y < xh(x)\}$$

Solution 5

i) We first note that the vertical line $x = 0$ does not intersect A at all, and hence the claim holds. Now consider the line $y = mx$, for some $m \in \mathbb{R}$. If $m \leq 0$, the line does not intersect A, and so the claim holds. Otherwise, note that that point $(t, mt) \in A$ if any only if $t > 0$ and $mt < t^2$. Thus, for $t < m$, $(t, mt) \notin A$. Hence, the interval

$$I = \{(t, mt) \mid t \in (-m, m)\}$$

does not intersection A.

ii) As x^2 is continuous and tends to 0 as x tends to 0, for any $\delta > 0$, we can find a point $\left(x, \frac{x^2}{2}\right) \in A$ such that

$$\| \left(x, \frac{x^2}{2}\right) \| < \delta$$

but $|f\left(x, \frac{x^2}{2}\right) - f(0,0)| = 1$. Thus, given any $0 < \varepsilon < 1$, we cannot find a δ such that the continuity condition holds at $(0,0)$.

iii) If $u = 0$, then $g(t) = 0$ for all t and is therefore continuous at 0. Otherwise, there exists $m \in \mathbb{R}$ such that $v = mu$. If $m \leq 0$, then $g(t) = 0$, and is therefore continuous. If $m > 0$, then for all $|t| < m$,

$$|g(t) - g(0)| = |0 - 0| = 0 < \varepsilon$$

for any $\varepsilon > 0$, as $g(t) \notin A$. Thus, $g(t)$ is continuous at 0.

iv) Note that, as $h(t)$ is continuous, $th(t) \to 0$ as $t \to 0$. Hence, for any $\delta > 0$, we can find $x > 0$ such that $\|(x, xh(x))\| < \delta$, but as $(x, xh(x)/2) \in A_h$, $f_h(x, xh(x)/2) = 1$ cannot approach $f(0,0) = 0$.

As for $g_h(t)$, it suffices to show that $g_h(t) = 0$ for t sufficiently smaller. If $u = 0$ or $v = mu$ with $m < 0$, then $g_h(t) = 0$ for all t, and we are done. If $v = mu$ with $m > 0$, then note that for all t such that $m > h(t)$, we have that $(tu, tv) \notin A_h$. As $h(t) \to 0$ as $t \to 0$ (by continuity of h), there exists some $\delta > 0$ such that $h(t) < m$ for all $|t| < \delta$. Thus, for all $|t| < \delta$, $g_h(t) = 0$, and so g_h is continuous at 0.