MAU22203/33203 - Analysis in Several Real Variables

Tutorial Sheet 1

Trinity College Dublin

Course homepage

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Convergence in* $\mathbb{R}$

(i) What is the least upper bound of the set $\{x \in \mathbb{R} \mid x^2 - 3x + 2 < 0\}$?

(ii) What is the greatest lower bound of the set $\{\sin(x) + \cos(x) \mid 0 \leq x \leq \pi\}$?

(iii) Define a sequence $\{x_n = \sum_{k=0}^{n} \frac{1}{k!}\}$. Is this monotonic? Bounded? Convergent?

(iv) Determine the limit (or argue that no such limit exists) of

$$\left\{ x_n = \frac{2^n - 2^{-n}}{3^n + 3^{-n}} \right\}.$$

Solution 1

(i) The quadratic equation $x^2 - 3x + 2$ has roots $x = 1$ and $x = 2$, and is easily seen to be positive for $x \notin [1, 2]$. Therefore the set

$$\{x \mid x^2 - 3x + 2 < 0\} = (1, 2)$$

which has least upper bound 2.

(ii) The function $f(x) = \sin(x) + \cos(x)$ is continuous and defined on a closed interval $[0, \pi]$, and therefore achieves its minimum on this interval. At $x = 0$ or $x = \pi$, $f(x) = \pm 1$, so it remains to check for any internal minima. f has turning points wherever $f'(x) = \cos(x) - \sin(x) = 0$. In the interval $(0, \pi)$, this holds only for $x = \frac{\pi}{4}$ and $x = \frac{3\pi}{4}$. Checking these, we see that $f(\frac{3\pi}{4}) = -\sqrt{2} < -1$, so the greatest lower bound is $-\sqrt{2}$.

(iii) The sequence $\{x_n\}$ is monotonically increasing, as $x_{n+1} = x_n + \frac{1}{(n+1)!}$ is obtained by adding a positive term. It is bounded above, since

$$n! \geq 2^{n-1} \Leftrightarrow \frac{1}{n!} \leq 2^{1-n}$$

for all $n \geq 1$ and so

$$x_n \leq \sum_{k=0}^{n} 2^{1-n} < 4$$

and bounded below by 0. Hence it is convergent (to e, but that's a different problem).

(iv) Note that the sequence is bounded below by 0, and that since

$$2^n - 2^{-n} < 2^n,$$
$$3^n + 3^{-n} > 3^n,$$

we must have that $x_n < \left(\frac{2}{3}\right)^n$, which tends to 0 as $n \to \infty$. Hence, we must have that $\lim_{n\to\infty} x_n = 0$.

Exercise 2 *Practice with norms*

(i) Show that, for $\vec{x}$, $\vec{y} \in \mathbb{R}^m$,

$$2\|\vec{x}\|^2 + 2\|\vec{y}\|^2 = \|\vec{x} + \vec{y}\|^2 + \|\vec{x} - \vec{y}\|^2.$$

(ii) Given $\vec{x}_1 \neq \vec{x}_2 \in \mathbb{R}^m$ and $0 < c < 1$, prove that there exists $\vec{y} \in \mathbb{R}^m$ and $r \in \mathbb{R}$ such that the sets

$$\{\vec{x} \in \mathbb{R}^m \mid \|\vec{x} - \vec{x}_1\| = c\|\vec{x} - \vec{x}_2\|\}$$

and

$$\{\vec{x} \in \mathbb{R}^m \mid \|\vec{x} - \vec{y}\| = r\}$$

are equal.

Solution 2

(i) Recall that $\|\vec{x}\|^2 = \langle x, x\rangle$. Thus

$$\begin{aligned}\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 =& \langle \vec{x}+\vec{y}, \vec{x}+\vec{y}\rangle + \langle \vec{x}-\vec{y}, \vec{x}-\vec{y}\rangle \\ =& \langle \vec{x}, \vec{x}\rangle + 2\langle \vec{x}, \vec{y}\rangle + \langle \vec{y}, \vec{y}\rangle \\ &+ \langle \vec{x}, \vec{x}\rangle - 2\langle \vec{x}, \vec{y}\rangle + \langle \vec{y}, \vec{y}\rangle \\ =& 2\|\vec{x}\|^2 + 2\|\vec{y}\|^2.\end{aligned}$$

(ii) Squaring the defining condition of the first set (which changes nothing, as all quantities are non-negative), gives us that

$$\langle \vec{x}-\vec{x}_1, \vec{x}-\vec{x}_1\rangle = c^2\langle \vec{x}-\vec{x}_2, \vec{x}-\vec{x}_2\rangle.$$

Expanding this out using bilinearity, and rearranging terms, we get

$$(1-c^2)\|\vec{x}\|^2 - 2\langle \vec{x}_1 - c^2\vec{x}_2, \vec{x}\rangle + \|\vec{x}_1\|^2 - c^2\|\vec{x}_2\|^2 = 0$$

Defining $\vec{y} := \frac{1}{1-c^2}\vec{x}_1 - \frac{c^2}{1-c^2}\vec{x}_2$, we note that

$$\begin{aligned}\|\vec{x}-\vec{y}\|^2 &= \|\vec{x}\|^2 - 2\langle \vec{y}, \vec{x}\rangle + \|\vec{y}\|^2 \\ &= \|\vec{y}\|^2 + \frac{c}{1-c^2}\|\vec{x}_2\|^2 - \frac{1}{1-c^2}\|\vec{x}_1\|^2\end{aligned}$$

as required, taking r^2 to be the right hand side. In order for r to be a real number, we need to check that the right hand side is non-negative, but this follows quickly by expanding out $\|\vec{y}\|^2$ to obtain that

$$r^2 = \frac{c^2}{(1-c^2)^2}\|\vec{x}_2\|^2 - \frac{2c^2}{(1-c^2)^2}\langle \vec{x}_2, \vec{x}_1\rangle + \frac{c^2}{(1-c^2)^2} = \frac{c^2\|\vec{x}_2 - \vec{x}_1\|^2}{(1-c^2)^2} \geq 0$$

Exercise 3 *Fun with Cauchy-Schwarz*

1. Show that for any $x_1, \ldots, x_m \in \mathbb{R}$,

$$(x_1 + x_2 + \cdots + x_m)^2 \leq m\left(x_1^2 + \cdots + x_m^2\right)$$

2. Show that, for any $x, y \in \mathbb{R}$

$$(x+y)^2 \leq (x^2+1)(y^2+1).$$

Solution 3

(i) This is precisely the CS inequality for $\vec{x} = (x_1, \ldots, x_m)$ and $\vec{y} = (1, \ldots, 1)$.

(ii) This is precisely the CS inequalty for $\vec{x} = (x, 1)$, $\vec{y} = (1, y)$.

Exercise 4 *Sequences in* $\mathbb{R}^m$

Determine the limit, or argue that it does not exist, of the following sequences

(i) $\{(x_{n,1}, x_{n,2}) = (\frac{1+n}{1-2n}, \frac{1}{n^3})\}$

(ii) $\{(x_{n,1}, x_{n,2}) = (1 - 2^{-n}, n\sin(n^{-1}))\}$

(iii) $\{(x_{n,1}, x_{n,2}) = \big((1 - n^{-1})\cos(\frac{2\pi n}{7}), (1 + n^{-1})\sin(\frac{2\pi n}{7})\big)\}$

In the above questions, you may compute one variable limits using any technique you like. A formal $\varepsilon - N$ *proof is not required.*

Solution 4

(i) In the first coordinate, we can divide above and below by n to get that

$$\lim_{n\to\infty} \frac{1+n}{1-2n} = \lim_{n\to\infty} \frac{\frac{1}{n}+1}{\frac{1}{n}-2} = -\frac{1}{2}.$$

The second coordinate is strictly decreasing and must tend to 0. Hence $\vec{x}_n \to (-\frac{1}{2}, 0)$.

(ii) As $2^{-n} \to 0$, the first coordinate tends to 1. As $\frac{\sin(x)}{x} \to 1$ as $x \to 0$, we must have that

$$n\sin(n^{-1}) = \frac{\sin(\frac{1}{n})}{\frac{1}{n}} \to 1$$

as n tends to ∞. Hence $\vec{x}_n \to (1, 1)$.

(iii) Note that, while $1 - n^{-1} \to 1$, $\cos(\frac{2\pi n}{7})$ cycles through 13 distinct values, and therefore the first component, and hence the sequence $\{\vec{x}_n\}$, cannot converge.