MAU22203/33203 - Analysis in Several Real Variables

Tutorial Sheet 4

Trinity College Dublin

Course homepage

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Computing derivatives*

i) Using limits, verify that the derivative of $f(x,y) = x^2 - 18xy + y^2$ is equal to the Jacobian at all points (p,q).

ii) Define a map $\varphi : \mathbb{R}^8 \to \mathbb{R}^4$ via matrix multiplication: if

$$\varphi(\vec{x}) = (f_1(\vec{x}), f_2(\vec{x}), f_3(\vec{x}), f_4(\vec{x}))$$

the components are determined by

$$\begin{pmatrix} f_1(\vec{x}) & f_3(\vec{x}) \\ f_2(\vec{2}) & f_4(\vec{4}) \end{pmatrix} = \begin{pmatrix} x_1 & x_3 \\ x_2 & x_4 \end{pmatrix} \begin{pmatrix} x_5 & x_7 \\ x_6 & x_8 \end{pmatrix}.$$

Determine $(\mathrm{D}\,\varphi)_{\vec{x}}$.

iii) Hence show that, viewing (2×2)-matrices as elements of $\mathbb{R}^4$,

$$(\mathrm{D}(AB))_t = (\mathrm{D}\,A)_t B(t) + A(t)(\mathrm{D}\,B)_t$$

for any differentiable function $A, B : \mathbb{R} \to \mathbb{R}^4$.

Exercise 2 *Inverse Function Theorem*

i) Call a continuously differentiable function $f : \mathbb{R}^m \to \mathbb{R}^m$ locally invertible at $\vec{p}$ if there exists an open set $U \subset \mathbb{R}^m$ containing $\vec{p}$ such that $f : U \to f(U)$ is a bijection with a continuously differentiable inverse. Is

$$f(x, y) = \begin{pmatrix} e^{xy} \\ \sin(y + x) \end{pmatrix}$$

locally invertible at $(0, \pi)$?

ii) Let A be the set of $(x, y) \in \mathbb{R}^2$ such that f is locally invertible at (x, y). Is A open, closed, or neither in $\mathbb{R}^2$?

Exercise 3 *The implicit function theorem*

i) Let

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

Show that, for every $(x_0, y_0) \in S^1$, there exists open $V \subset \mathbb{R}^2$ such that $V \cap S^1$ is homeomorphic to an open $U \subset \mathbb{R}$.

ii) Show that there exists open $V \subset \mathbb{R}^2$ such that $V \cap S^1$ is homeomorphic to an open interval $(a, b) \subset \mathbb{R}$.

Hint: Think about connectedness

iii) Let

$$A = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2,\ x + y + z = 0\}.$$

Show that, for all but finitely many $(x, y, z) \in A$, there exists open $V \subset \mathbb{R}^3$ such that $V \cap A$ is homeomorphic to an open $U \subset \mathbb{R}$.

iv) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. Suppose that for each $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $f(x, y) = 0$. Denote this y by $c(x)$. Suppose further that $\partial_y f(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$. Show that c is differentiable, with derivative

$$c'(x) = -\frac{\partial_x f(x, c(x))}{\partial_y f(x, c(x)}.$$

Exercise 4 *The Challenge*

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function with $|f'(x)| \leq c < 1$ for all $x \in \mathbb{R}$. Define $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\phi(x, y) = (x + f(y), y + f(x)).$$

1. Show that ϕ is continuously differentiable
2. Show the conditions of the inverse function theorem apply, and hence ϕ has a local inverse $\mu_{(p,q)}$ near every point $(p, q) \in \phi(\mathbb{R}^2)$
3. Show that ϕ is injective and therefore there exists $\mu : \phi(\mathbb{R}^2) \to \mathbb{R}^2$ such that $\phi(\mu(p, q)) = (p, q)$.

   Hint: Suppose otherwise and find a way to use the mean value theorem
4. Show that $\phi(\mathbb{R}^2)$ is open and explain why μ must be continuously differentiable.
5. Show that ϕ is surjective.

   Hint: can you rewrite surjectivity as a fixed point condition, or show the image is closed?