

MAU22203/33203 - Analysis in Several Real Variables

Tutorial Sheet 3

Trinity College Dublin

Course homepage

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Mean value theorem*

- i) Does the mean value theorem apply to the function $f(x) = 4\sqrt{x}$ on $[0, 1]$? If so, determine a value of c satisfying the conclusions of the theorem.
- ii) A man is driving in a 80km/h zone. He enters the zone at 13:00, and has travelled 405km in a single direction by 18:00. Did he obey the speed limit at all times?
- iii) Let $f : [a, b] \rightarrow \mathbb{R}$ be a Riemann integrable function. Show that there exists $c \in (a, b)$ such that

$$\int_a^b f(x)dx = f(c)(b - a).$$

Exercise 2 *Differentiability and first order approximation*

- i) Compute the partial derivatives of $f(x, y) = \frac{x^3 y}{1+x^2+y^4}$.
- ii) Compute the partial derivatives of $f(x, y) = x^y$ at a point where $x > 0$ and $y > 0$.
- iii) Compute the partial derivatives of $f(x, y) = \sin(x \sin(y))$.
- iv) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function. Show that f is differentiable at x_0 if and only if there exist $a, b \in \mathbb{R}$ such that

$$\lim_{h \rightarrow 0} \frac{f(x_0 + h) - a - bh}{h} = 0.$$

Exercise 3 *Mixed partial derivatives are harder than you thin*

Given a function $f : \mathbb{R}^m \rightarrow \mathbb{R}$ with first order partial derivatives, we can attempt to define second order partial derivatives by

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x}) = (\partial_i \partial_j f)(\vec{x}) := \lim_{h \rightarrow 0} \frac{(\partial_j f)(\vec{x} + h \vec{e}_i) - (\partial_j f)(\vec{x})}{h}$$

whenever this limit exists. In more standard notation

$$\frac{\partial^2 f}{\partial x_i \partial x_j} := \frac{\partial}{\partial x_i} \left(\frac{\partial f}{\partial x_j} \right).$$

When the first order partial derivatives are continuous, if the second order partial derivatives, when they are defined, are also continuous, the order in which we take the derivatives does not matter. In general

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \neq \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

Define a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) := \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

- i) Compute the partial derivatives of f with respect to x and y away from $(0, 0)$, using the standard rules, and at $(0, 0)$, using the limit definition.
- ii) Compute $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ away from $(0, 0)$ and at $(0, 0)$, using the limit definition if necessary.

Exercise 4 *Matrix norms and exponentials*

- i) Let $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a linear map. Show that the map $\vec{x} \rightarrow T\vec{x}$ is continuous at every point of $\mathbb{R}^m$.

Hint: Recall the first homework. Consider the zero map as a separate case

- ii) Define the Hilbert-Schmidt norm of a real $(m \times n)$ -matrix $A = (A_{i,j})$ by

$$\|A\|_{HS} = \sqrt{\sum_{i=1}^m \sum_{j=1}^n A_{i,j}^2}.$$

Show that the Hilbert-Schmidt norm satisfies the triangle inequality:

$$\|A + B\|_{HS} \leq \|A\|_{HS} + \|B\|_{HS}.$$

Hint: Either replicate the proof for the Euclidean norm, or figure out a way to view this as a Euclidean norm

- iii) Let T be an $(n \times n)$ -matrix. Define a sequence of matrices by $\{T_s = \sum_{k=0}^s \frac{1}{k!} T^k\}$. Viewing this as a sequence in $\mathbb{R}^{n^2}$, show that this is a Cauchy sequence and hence converges. You may freely use the fact that

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

converges for all real $\mathbb{R}$.

- iv) Define $\exp(T) : \lim_{s \rightarrow \infty} T_s$ as the limit of this sequence. Can you find an example of two square matrices A, B such that

$$\exp(A + B) \neq \exp(A) \exp(B)?$$