MAU22203/33203 - Analysis in Several Real Variables

Tutorial Sheet 2

Trinity College Dublin

Course homepage

The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Convergent subsequences*

Let $\{\vec{x}_k\}$ be a bounded sequence in $\mathbb{R}^m$ that does not converge to $\vec{0}$.

i) Show that there exist $R, c > 0$ such that $\{\vec{x}_k\}$ contains a subsequence contained within the set

$$\{\vec{x} \in \mathbb{R}^m \mid c \leq \|\vec{x}\| \leq R\},$$

ii) Show that this subsequence contains a convergent subsequence, converging to a point not equal to $\vec{0}$,

iii) Hence conclude that $\{\vec{x}_k\}$ contains a convergent subsequences, converging to a point other than $\vec{0}$.

Exercise 2 *Interiors*

Given a subset $X \subset \mathbb{R}^m$, we define its interior X^o, closure $\overline{X}$, and boundary ∂X by

$$X^o = \bigcup_{\substack{U \subset X \\ U \text{ open}}} U,$$
$$\overline{X} = \bigcap_{\substack{F \supset X \\ F \text{ closed}}} F,$$
$$\partial X = \overline{X} \setminus X^o.$$

i) Show that X^o is open and $\overline{X}$ is closed,

ii) Find the interior, closure, and boundary of

$$\{\vec{x} \in \mathbb{R}^m \mid \|\vec{x}\| \leq 2\},$$

iii) Find the interior, closure, and boundary of

$$\{(x, y) \in \mathbb{R}^2 \mid x \geq 2, y > 0\},$$

iv) Find the interior, closure, and boundary of

$$\{x \in \mathbb{R} \mid x \in \mathbb{Q}\}.$$

Hint: It might be easier to consider the interior of the complement.

Exercise 3 *Topological stuff*

i) Identify whether the following sets are open/closed in $\mathbb{R}^2$:

a) $\{(x, y) \in \mathbb{R}^2 \mid xy = 1\}$,

b) $\{(x, y) \in \mathbb{R}^2 \mid x \neq 0, y \neq 0\}$,

c) $\{(x, y) \in \mathbb{R}^2 \mid \max\{|x|, |y|\} = 1\}$,

d) $\{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 < 17xy < 34\}$

ii) Suppose that $X \subset \mathbb{R}^m$ is not closed. Show that there exists a point $\vec{v} \in \mathbb{R}^m \setminus X$ such that

$$B(\vec{v}, \varepsilon) \cap X \neq \emptyset$$

for all $\varepsilon > 0$.

iii) Call a subset A of $\mathbb{Z}$ open if $\mathbb{Z} \setminus A$ is finite or if A is empty. Show that this collection of open sets defines a topology on $\mathbb{Z}$.

Exercise 4 *Continuous functions and convergent sequences*

i) Without resorting to an $\varepsilon - \delta$ proof, show that if $\phi_1, \phi_2 : \mathbb{R}^m \to \mathbb{R}^n$ are continuous functions, then

$$\vec{x} \mapsto \phi_1(\vec{x}) + \phi_2(\vec{x})$$

is a continuous function.

ii) Without resorting to an $\varepsilon - N$ argument, prove that if we have two sequences $\{\vec{x}_k\}$ and $\{\vec{y}_k\}$ in $\mathbb{R}^m$ converging to $\vec{p}$ and $\vec{q}$ respectively, then the sequence in $\mathbb{R}$ given by

$$\{z_k = \langle \vec{x}_k, \vec{y}_k \rangle\}$$

converges to $\langle \vec{p}, \vec{q} \rangle$.

iii) Let $X \subset \mathbb{R}^m$ be a strict subset of $\mathbb{R}^m$, and suppose we have a point $\vec{p} \in \mathbb{R}^m \setminus X$. Show that

$$\phi_{\vec{p}}(\vec{x}) = \frac{1}{\|\vec{x} - \vec{p}\|}$$

is continuous on X.

iv) Hence argue that if X is not closed in $\mathbb{R}^m$, there exists $\vec{p} \in \mathbb{R}^{\setminus} X$ such that the image $\phi_{\vec{p}}(X)$ is not bounded in $\mathbb{R}$.

Hint: Use 3.ii to construct a bad sequence.

Exercise 5 *A challenge from Spivak*

The following is a neat problem, where drawing pictures should make the answer obvious, though writing a good argument for it could be hard. Define a set $A \subset \mathbb{R}^2$ by

$$A = \{(x, y) \in \mathbb{R}^2 \mid 0 < x,\ 0 < y < x^2\}$$

i) Show that every straight line in $\mathbb{R}^2$ passing through $(0, 0)$ contains an open interval I such that $I \cap A = \emptyset$.

ii) Define

$$f(x, y) = \begin{cases} 1 \text{ if } (x, y) \in A, \\ 0 \text{ if } (x, y) \notin A. \end{cases}$$

Show that $f(x, y)$ is not continuous at $(0, 0)$.

iii) Fix a vector $(u, v) \in \mathbb{R}^2$ and define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(t) := f(tu, tv).$$

Show that g is continuous at 0.

iv) Show that, for any continuous $h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, the function

$$f_h(x, y) = \begin{cases} 1 \text{ if } (x, y) \in A_h, \\ 0 \text{ if } (x, y) \notin A_h. \end{cases}$$

is not continuous at $(0, 0)$, while the function $g_h(t) = f_h(tu, tv)$ is continuous at 0. Here, we define

$$A_h := \{(x, y) \in \mathbb{R}^2 \mid 0 < x,\ 0 < y < xh(x)\}$$