# MAU22203/33203 - Analysis in Several Real Variables

Exercise Sheet 4

Trinity College Dublin

Course homepage

This is an entirely optional homework. If submitted, the best 3 out of 4 homeworks will be considered for your continuous assessment. Answers are due for December $7^{\text{th}}$, 23:59.

**Exercise 1** *Existence is enough right? (60pts)*

i) (15pts) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show there exists $c \in (0,1)$ such that

$$\int_0^1 f(x),\ dx = f(c)$$

*Hint: Mean value theorem*

ii) (20pts) Let $F : [0,1]^2 \to \mathbb{R}$ be a continuous function. Show there exists $(c,d) \in (0,1)^2$ such that

$$\int_0^1 \int_0^1 F(x,y)\, dx\, dy = F(c,d)$$

*Hint: Do it in steps. Don't forget to check for continuity along the way!*

iii) (25pts) Let $F(x,y) = x^2 + xy + y^2$. Determine a point $(c,d) \in (0,1)^2$ such that

$$\int_0^1 \int_0^1 F(x,y)\, dx\, dy = F(c,d)$$

*Hint: I don't think there is a smart way to do this, but there is a systematic way.*

## **Exercise 2** *Applying Fubini's theorem (40pts)*

1. (10pts) Let $f : [0,1]^3 \to \mathbb{R}$ be a continuous function. Using Fubini's theorem for 2 variable functions, show that

$$\int_0^1 \int_0^1 \int_0^1 f(x,y,z)\,dx\,dy\,dz = \int_0^1 \int_0^1 \int_0^1 f(x,y,z)\,dy\,dz\,dx$$

*Hint: Remember the integrand must be continuous to use Fubini!*

2. (10 pts) Show that

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)^2}{x^2+y^2} \text{ if } (x,y) \neq (0,0), \\ 0 \text{ otherwise} \end{cases}$$

is continuous on $[0,1]^2$.

*Hint: Bound the absolute value near 0 in terms of norms. Note that* $|x^2 - y^2| \leq x^2 + y^2$.

3. (20 pts) Compute

$$\int_{[0,1]^2} \frac{xy(x^2-y^2)^2}{x^2+y^2}\,dA$$

*Hint: You may (and probably should should) use without proof that*

$$\int_0^1 t^5 \left(\ln(t^2+1) - \ln(t^2)\right)\,dt = \frac{\ln(2)}{3} - \frac{1}{12}$$