MAU22203/33203 - Analysis in Several Real Variables

Exercise Sheet 3

Trinity College Dublin

Course homepage

Answers are due for November 30$^{\text{th}}$, 23:59.
The use of electronic calculators and computer algebra software is allowed.
If typesetting your work, the following code can be used to product a 3×4 matrix, and can be easily modified to produce other sizes of matrices:

```
\[ A= \begin{pmatrix}
    a & b & c & d \\
    e & f & g & h \\
    i & j & k & l
    \end{pmatrix}
\]
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Exercise 1 *Applications of the inverse function theorem (40pts)*

i) (15pts) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map

$$f(x, y) = \begin{pmatrix} ye^x \\ \cos(xy) \end{pmatrix}$$

Determine the derivative $(\mathrm{D} f)_{(x,y)}$ at all points at which f is differentiable.

ii) (15pts) We say that f is locally invertible at $(x, y) \in \mathbb{R}^2$ if there exists open $U \ni (x, y)$ such that $f : U \to f(U)$ has a continuously differentiable inverse. Let A be the set of points (x, y) at which f is locally invertible. Is A open, closed, or either?

Hint: Can we write A as the inverse image of a set under a continuous map?

iii) (10pts) Let $(x_0, y_0) \in A$. What is the derivative of the local inverse of f at $f(x_0, y_0)$?

Exercise 2 *Applying the implicit function theorem (60pts)*

1. (20pts) Let
$$S = \{(x, y) \in \mathbb{R}^2 \mid 8y^2 = x^3 + 7x^2\}$$
Let S_0 be the set of points $(x_0, y_0) \in S$ for which the assumptions of the implicit function theorem apply. Determine S_0. Is S_0 open or closed (or neither) in S?

2. (10 pts) Explain why, for open $V = B((0, 0), 1)$, there does not exist an open interval (a, b) such that $V \cap S$ is homeomorphic to (a, b). You do not have to give a rigorous proof.

   Hint: Look at the graph and count the number of points in the boundaries. As a reminder, a homeomorphism is a continuous bijection with a continuous inverse. It preserve all topological properties, like openness, closedness, as well as Cauchy sequences and most notions of boundaries.

3. (30 pts) Determine the maximal open $U \subset S$ containing $(x_0, y_0) = (-7, 0)$ on which x can be expressed as a continuously differentiable function of y.

   Hint: Implicit function theorem tells you where you can do this locally. Is it possible to patch these local functions together? What do the local functions look like on the intersection of their domains? You may use without proof that if $(x, y) \in S$, $x \geq -7$.