MAU22203/33203 - Analysis in Several Real Variables

Exercise Sheet 2

Trinity College Dublin

Course homepage

Answers are due for November 9^{th}, 23:59.
The use of electronic calculators and computer algebra software is allowed, though reasonably thorough computations are expected in Exercise 1, i.e. present the differentiation, though you can simplify using a computer.

Exercise 1 *Lines are not enough (60 pts)*

For a function $f : \mathbb{R} \to \mathbb{R}$, differentiability implies continuity. As we will see here, the optimistic analogue for functions $f : \mathbb{R}^m \to \mathbb{R}$ does not hold, no matter how nice our function is along a line. Consider the function

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto \begin{cases} \frac{2x^3y}{x^6+y^2} \text{ if } (x, y) \neq (0, 0), \\ 0 \text{ if } (x, y) = (0, 0). \end{cases} \end{aligned}$$

1. (20 pts) Show the restriction of f to any line through the origin is continuous at $(0, 0)$ i.e.

$$\lim_{t \to 0} f(ta, tb) = 0$$

for any $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$. You do not need to give an $\varepsilon - \delta$ proof, but be sure to justify why this limit exists.

Hint: Consider the case $b = 0$ separately.

2. (25 pts)Show that the directional derivative of f along a unit vector $\vec{v} = (u, v)$ is 0 at 0: $\partial_{\vec{v}} f(0,0) = 0$.

3. (15 pts) Show that despite f being continuous along every line, with a defined derivative in every direction, f is not continuous as a function of two variables at $(0,0)$

   Hint: Consider $\lim_{t\to 0} f(t, g(t))$ for some $g(t)$ a polynomial in t that vanishes at $t = 0$. What $g(t)$ causes the denominator to have a higher order of vanishing than the numerator?

Exercise 2 *Deriving a special case of Lagrange multipliers (40pts)*

For purposes of this question, you should pretend that you have no pre-existing knowledge of Lagrange multipliers or optimisation with constraints. You may, however, freely use chain rule for functions of several variables.

1. (30 pts) Let
$$H^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 = 1\}$$
   be the standard hyperbola, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function with first order partial derivatives. Suppose that f, restricted to a function on the hyperbola, has a local extremum at a point $(x_0, y_0) \neq (1, 0)$. By writing
$$H^1 = \{(\cosh(t), \sinh(t)) \mid t \in \mathbb{R}\}$$
   show that there exists $\lambda \in \mathbb{R}$ such that
$$\frac{\partial f}{\partial x}(x_0, y_0) = 2x_0\lambda, \quad \frac{\partial f}{\partial y}(x_0, y_0) = -2y_0\lambda.$$

   Hint: Since $x_0, y_0 \neq 0$, we can define λ satisfying one of these equations. Consider the function $F(t) = f(\cosh(t), \sinh(t))$. Freely using chain rule for such compositions, what does the derivative of F look like at a local extremum? Can we show that this implies λ satisfies the other?

2. (10 pts) Using this determine the possible local extrema of $f(x, y) = x^3 + 2y^3$ restricted to $x^2 - y^2 = 1$, and the associated λ.

   Remark: This means without using the parameterisation of H^1 in terms of hyperbolic trigonometric functions!