MAU22203/33203 - Analysis in Several Real Variables

Exercise Sheet 1

Trinity College Dublin

Course homepage

Answers are due for October 12th, 23:59.
The use of electronic calculators and computer algebra software is allowed.

Exercise 1 *Properties of sequences (60pts)*

Let $\{\vec{x}_n\}$ and $\{\vec{y}_n\}$ be two sequences of points in $\mathbb{R}^m$, and let $\lambda \in \mathbb{R}$ be a real number. Suppose that $\{\vec{x}_n\}$ converges to a point $\vec{p}$, and $\{\vec{y}_n\}$ converges to a point $\vec{q}$. By giving a formal ε-N proof, establish the following:

(15pts) The sequence $\{\lambda\vec{x}_n\}$ converges to $\lambda\vec{p}$,

Hint: consider $\lambda = 0$ as a separate case.

(15pts) The sequence $\{\vec{z}_n = \vec{x}_n + \vec{y}_n\}$ converges to $\vec{p} + \vec{q}$.

(30pts) Suppose further that, for all $n > 0$, $\|\vec{x}_n - \vec{y}_n\| < \frac{1}{n}$. Conclude that $\vec{p} = \vec{q}$.

Hint: We haven't yet shown that limits commute with norms, so we can't freely conclude this. Can we show that $\|\vec{p} - \vec{q}\|$ must be smaller than all positive reals? Try the extending the triangle inequality to a sum of three terms!

Exercise 2 *Bounded operators (40pts)*

In the following, you may use any standard facts from your first year courses. Let $A : \mathbb{R}^m \to \mathbb{R}^m$ be a linear transformation represented by the $(m \times m)$-matrix $(A_{i,j})_{\substack{1\le i\le m\\ 1\le j\le m}}$ with respect to the standard bases.

(15 pts) Show that there exists a constant $C > 0$ such that

$$\|A\vec{x}\| \le C\|\vec{x}|$$

for all $\vec{x}\| \in \mathbb{R}^m$.

Hint: what are the components of $A\vec{x}$ and how could we bound them using the Cauchy-Schwarz inequality?

(10 pts) Hence, or otherwise, show that

$$\|A^n\vec{x}\| \le C^n\|\vec{x}\|$$

for all $\vec{x} \in \mathbb{R}^m$ and all $n \ge 0$.

(15 pts) Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the below matrix, and define a sequence of points in $\mathbb{R}^3$ by $\vec{x}_n := A^{n-1}\vec{x}_1$, where $\vec{x}_1$ is given below. Prove that $\{\vec{x}_n\}$ converges to $\vec{0}$.

$$A = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{5} & 0 & \frac{1}{20} \end{pmatrix}, \quad \vec{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Hint: Try to bound $\|\vec{x}_n - \vec{0}\| = \|\vec{x}_n\|$ by something that converges to 0.