MAU22203/33203 - Analysis in Several Real Variables

Exercise Sheet 4

Trinity College Dublin

Course homepage

This is an entirely optional homework. If submitted, the best 3 out of 4 homeworks will be considered for your continuous assessment. Answers are due for December 7^{th}, 23:59.

Exercise 1 *Existence is enough right? (60pts)*

i) (15pts) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show there exists $c \in (0,1)$ such that

$$\int_0^1 f(x),\ dx = f(c)$$

Hint: Mean value theorem

ii) (20pts) Let $F : [0,1]^2 \to \mathbb{R}$ be a continuous function. Show there exists $(c,d) \in (0,1)^2$ such that

$$\int_0^1 \int_0^1 F(x,y)\, dx\, dy = F(c,d)$$

Hint: Do it in steps. Don't forget to check for continuity along the way!

iii) (25pts) Let $F(x,y) = x^2 + xy + y^2$. Determine a point $(c,d) \in (0,1)^2$ such that

$$\int_0^1 \int_0^1 F(x,y)\, dx\, dy = F(c,d)$$

Hint: I don't think there is a smart way to do this, but there is a systematic way.

Solution 1

i) Denote by $F : \mathbb{R} \to \mathbb{R}$ the function

$$F(x) = \int_0^x f(t)\,dt$$

which is differentiable with derivative $f(x)$, by the fundamental theorem of calculus. It is continuous at 0 and 1, so we can apply the mean value theorem to conclude that there exists $c \in (0,1)$ such that

$$\frac{F(1) - F(0)}{1 - 0} = F'(c) = f(c)$$

We have that $F(0) = 0$ and $F(1) = \int_0^1 f(x)\,dx$, so the claim follows.

ii) Fixing y, the function $F(x,y)$ is continuous as a function of x, and so there exists $c_y \in (0,1)$ such that

$$\int_0^1 F(x,y)\,dx = F(c_y, y).$$

The left hand side is a continuous function of y, and so the right hand side is a continuous function of y. Hence, we can repeat the argument to show that there exists $d \in (0,1)$ such that

$$\int_0^1 \int_0^1 F(x,y)\,dx\,dy = \int_0^1 F(c_y, y)\,dy = F(c_d, d) = F(c,d)$$

where we write $c = c_d$.

Alternatively, note that $g(y) = \int_0^1 F(x,y)\,dx$ is continuous and hence there exists $d \in (0,1)$ such that

$$\int_0^1 F(x,d)\,dx = g(d) = \int_0^1 g(y)\,dy = \int_0^1 \int_0^1 F(x,y)\,dx\,dy$$

Similarly, $F(x,d)$ is a continuous function of x, so there exists $c \in (0,1)$ such that

$$F(c,d) = \int_0^1 F(x,d)\,dx = \int_0^1 \int_0^1 F(x,y)\,dx\,dy$$

iii) One approach is to take $c = \frac{1}{2}$ and compute

$$\frac{1}{4} + \frac{d}{2} + d^2 = F(c,d) = \int_0^1 \int_0^1 F(x,y)\,dx\,dy = \frac{11}{12}$$

Solving this for d, we find

$$d = \frac{\sqrt{105} - 3}{12} \in (0,1).$$

If we wanted to do this is a slightly more systematic way, we can split the computation into a few steps. We compute

$$c^2 + cy + y^2 = F(c,y) = \int_0^1 F(x,y)\,dx = \frac{1}{3} + \frac{y}{2} + y^2$$

and so to get $c \in (0,1)$, we must have

$$c(y) = -\frac{y}{2} + \frac{\sqrt{y^2 + 2y + 4/3}}{2}$$

as a function of y. Then, we compute

$$\frac{1}{3} + \frac{d}{2} + d^2 = F(c(d),d) = \int_0^1 \int_0^1 F(x,y)\,dx\,dy = \frac{11}{12}$$

and hence

$$d = \frac{\sqrt{93} - 3}{12}, \quad c = \frac{1}{8}\left(1 - \sqrt{\frac{31}{3}} + \sqrt{\frac{37}{3} + \frac{\sqrt{93}}{2}}\right).$$

I think.

Exercise 2 *Applying Fubini's theorem (40pts)*

1. (10pts) Let $f : [0,1]^3 \to \mathbb{R}$ be a continuous function. Using Fubini's theorem for 2 variable functions, show that

$$\int_0^1 \int_0^1 \int_0^1 f(x,y,z)\,dx\,dy\,dz = \int_0^1 \int_0^1 \int_0^1 f(x,y,z)\,dy\,dz\,dx$$

Hint: Remember the integrand must be continuous to use Fubini!

2. (10 pts) Show that

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)^2}{x^2+y^2} \text{ if } (x,y) \neq (0,0), \\ 0 \text{ otherwise} \end{cases}$$

is continuous on $[0,1]^2$.

Hint: Bound the absolute value near 0 in terms of norms. Note that $|x^2 - y^2| \leq x^2 + y^2$.

3. (20 pts) Compute

$$\int_{[0,1]^2} \frac{xy(x^2-y^2)^2}{x^2+y^2} \, dA$$

Hint: You may (and probably should should) use without proof that

$$\int_0^1 t^5 \left(\ln(t^2+1) - \ln(t^2)\right) \, dt = \frac{\ln(2)}{3} - \frac{1}{12}$$

Solution 2

i) For fixed z, the function $f(x,y,z)$ is continuous as a function of x and y. Hence

$$\int_0^1 \int_0^1 f(x,y,z) \, dx \, dy = \int_0^1 \int_0^1 f(x,y,z) \, dy \, dx$$

and so

$$\begin{aligned} \int_0^1 \int_0^1 \int_0^1 f(x,y,z) \, dx \, dy \, dz &= \int_0^1 \left(\int_0^1 \int_0^1 f(x,y,z) \, dx \, dy \right) \, dz \\ &= \int_0^1 \left(\int_0^1 \int_0^1 f(x,y,z) \, dy \, dx \right) \, dz \\ &= \int_0^1 \int_0^1 \int_0^1 f(x,y,z) \, dy \, dx \, dz. \end{aligned}$$

As $\int_0^1 f(x,y,z)\,dy$ is a continuous function of x and z, we have that

$$\begin{aligned}\int_0^1\int_0^1\int_0^1 f(x,y,z)\,dy\,dx\,dz &= \int_0^1\left(\int_0^1\int_0^1 f(x,y,z)\,dy\right)\,dx\,dz\\ &= \int_0^1\left(\int_0^1\int_0^1 f(x,y,z)\,dy\right)\,dz,dx\\ &= \int_0^1\int_0^1\int_0^1 f(x,y,z)\,dy\,dz\,dx.\end{aligned}$$

ii) The function is clearly continuous away from $(0,0)$ so it suffices to show that

$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0.$$

Note that, for $(x,y)\neq(0,0)$

$$\begin{aligned}|f(x,y)| &= |xy|\frac{|x^2-y^2|^2}{|x^2+y^2|}\\ &\le |xy|\frac{(|x|^2+|y|^2)^2}{|x^2+y^2|}\\ &= |xy|frac(x^2+y^2)^2x^2+y^2\\ &= |xy|(x^2+y^2) = |xy|\cdot\|(x,y)\|^2\end{aligned}$$

which clearly tends to 0 as $(x,y)\to(0,0)$. Thus,

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

and f is continuous.

iii) Since f is continuous, we can compute this as the iterated integral

$$L = \int_0^1\int_0^1 \frac{xy(x^2-y^2)^2}{x^2+y^2}\,dx\,dy.$$

Letting $u = x^2 + y^2$ in the x-integral, we find that

$$\begin{aligned}
L &= \frac{1}{2}\int_0^1 \int_{y^2}^{y^2+1} \frac{y(u-2y^2)^2}{u}\,du\,dy \\
&= \frac{1}{2}\int_0^1 \int_{y^2}^{y^2+1} yu - 4y^3 + \frac{4y^5}{u}\,du\,dy \\
&= \frac{1}{4}\int_0^1 y - 6y^3 + 8y^5\ln(y^2+1) - 8y^5\ln(y^2)\,dy \\
&= \frac{1}{4}\left(\frac{1}{2} - \frac{3}{2} + \frac{8\ln(2)}{3} - \frac{2}{3}\right) \\
&= \frac{2\ln(2)}{3} - \frac{5}{12}
\end{aligned}$$