MAU22203/33203 - Analysis in Several Real Variables

Exercise Sheet 3

Trinity College Dublin

Course homepage

Answers are due for November 30^{th}, 23:59.
The use of electronic calculators and computer algebra software is allowed.
If typesetting your work, the following code can be used to product a 3×4 matrix, and can be easily modified to produce other sizes of matrices:

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\[ A= \begin{pmatrix}
    a & b & c & d \\
    e & f & g & h \\
    i & j & k & l
    \end{pmatrix}
\]
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Exercise 1 *Applications of the inverse function theorem (40pts)*

i) (15pts) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map

$$f(x,y) = \begin{pmatrix} ye^x \\ \cos(xy) \end{pmatrix}$$

Determine the derivative $(\mathrm{D}\, f)_{(x,y)}$ at all points at which f is differentiable.

ii) (15pts) We say that f is locally invertible at $(x,y) \in \mathbb{R}^2$ if there exists open $U \ni (x,y)$ such that $f : U \to f(U)$ has a continuously differentiable inverse. Let A be the set of points (x, y) at which f is locally invertible. Is A open, closed, or either?

Hint: Can we write A as the inverse image of a set under a continuous map?

iii) (10pts) Let $(x_0, y_0) \in A$. What is the derivative of the local inverse of f at $f(x_0, y_0)$?

Solution 1

i) We first compute the Jacobian

$$(\mathrm{J}\, f)_{(x,y)} = \begin{pmatrix} ye^x & e^x \\ -y\sin(xy) & -x\sin(xy) \end{pmatrix}$$

As all entries of the Jacobian are continuous on $\mathbb{R}^2$, f is differentiable everywhere in $\mathbb{R}^2$ with derivative $(\mathrm{D}\, f)_{(x,y)} = (\mathrm{J}\, f)_{(x,y)}$.

ii) Since f is continuously differentiable, the inverse function theorem tells us that f is locally invertible when it has invertible derivative. The derivative is invertible when it has non-zero determinant, and so

$$A = \{(x, y) \mid \det(\mathrm{D}\, f)_{(x,y)} \neq 0\}.$$

If the determinant is a continuous function of (x, y), then this set is open, as it is the inverse image of the open set $\mathbb{R} \setminus \{0\}$. We can check that

$$\det(\mathrm{D}\, f)_{(x,y)} = e^x \sin(xy)(y - xy)$$

is definitely continuous, so A is open.

iii) The derivative of the local inverse is given by the inverse of the derivative, and hence is equal to

$$\frac{1}{e^x \sin(xy)(y - xy)} \begin{pmatrix} -x\sin(xy) & -e^x \\ y\sin(xy) & ye^x \end{pmatrix}$$

Exercise 2 *Applying the implicit function theorem (60pts)*

1. (20pts) Let
$$S = \{(x, y) \in \mathbb{R}^2 \mid 8y^2 = x^3 + 7x^2\}$$
Let S_0 be the set of points $(x_0, y_0) \in S$ for which the assumptions of the implicit function theorem apply. Determine S_0. Is S_0 open or closed (or neither) in S?

2. (10 pts) Explain why, for open $V = B((0, 0), 1)$, there does not exist an open interval (a, b) such that $V \cap S$ is homeomorphic to (a, b). You do not have to give a rigorous proof.

   Hint: Look at the graph and count the number of points in the boundaries. As a reminder, a homeomorphism is a continuous bijection with a continuous inverse. It preserve all topological properties, like openness, closedness, as well as Cauchy sequences and most notions of boundaries.

3. (30 pts) Determine the maximal open $U \subset S$ containing $(x_0, y_0) = (-7, 0)$ on which x can be expressed as a continuously differentiable function of y.

   Hint: Implicit function theorem tells you where you can do this locally. Is it possible to patch these local functions together? What do the local functions look like on the intersection of their domains? You may use without proof that if $(x, y) \in S$, $x \geq -7$.

Solution 2

i) The function $f(x, y) = 8y^2 - x^3 - 7x^2$ is continuously differentiable with derivative
$$(\mathrm{J}\, f)_{(x_0,y_0)} = \begin{pmatrix} -3x^2 - 14x, 16y \end{pmatrix}$$
which has rank 1 unless
$$3x^2 + 7x = 16y = 0 \text{ which implies that} x \in \{0, \frac{-14}{3}\},\ y = 0$$
Since $f(\frac{-7}{3}, 0) \neq 0$, $(0, 0)$ is the only point in S such that $(\mathrm{J}\, f)_{(x_0,y_0)}$ has rank 0. Thus, $S_0 = S \setminus \{(0, 0)\}$. Furthermore, since $U = \mathbb{R}^2 \setminus \{(0, 0)\}$ is open in $\mathbb{R}^2$, $S_0 = S \cap U$ is open in S.

ii) Looking at the graph of of $f(x, y) = 0$, we see that $V \cap S$ is roughly a cross, which clearly has 4 boundary points, which an interval in $\mathbb{R}$ has only 2 boundary points. A homeomorphism would preserve the number of boundary points as it preserves all topological information, so no such homeomorphism can exist.

iii) Note that $3x^2 + 14x \neq 0$ for all $-7 \leq x \leq \frac{-14}{3}$, and hence x can be expressed locally as a function of y around every point of S satisfying this condition. Thus, we obtain an open $V_(x, y) \ni (x, y)$, open $U_{(x,y)}$ and a continuously differentiable $g_{(x,y)} : U_{(x,y)} \to \mathbb{R}$ such that

$$V_{(x,y)} \cap S = \{(g_{(x,y)}(z), z) \mid z \in U_{(x,y)}$$

If

$$V_{(x_1,y_1)} \cap V_{(x_2,y_2)} \neq \emptyset$$

then we must have that $g_{(x_1,y_1)} = g_{(x_2,y_2)}$ on the intersection, and so we can extend these to a single continuously differentiable function on their union. In this way, we can glue together all our local functions to obtain x as a continuously differentiable function of y on

$$U = \{(x, y) \in S \mid x < \frac{-14}{3}\}$$

which is open in S as it is the intersection of an open set with S.

We cannot extend this any further. We can either argue that the graphs

$$y = \pm\sqrt{\frac{x^3 + 14x}{8}}$$

have turning points at $x = \frac{-14}{3}$, and so the map $S \to \mathbb{R}$ taking $(x, y) \to x$ is not injective, so no g could exists, or we could use implicit differentiation to show that

$$\frac{dg}{dy} = \frac{\partial f}{\partial y}\frac{\partial x}{\partial f}$$

which is infinite at these points. A rigorous explanation is not needed - any valid heuristic will do.