MAU22203/33203 - Analysis in Several Real Variables

Exercise Sheet 2

Trinity College Dublin

Course homepage

Answers are due for November 9^{th}, 23:59.
The use of electronic calculators and computer algebra software is allowed, though reasonably thorough computations are expected in Exercise 1, i.e. present the differentiation, though you can simplify using a computer.

Exercise 1 *Lines are not enough (60 pts)*

For a function $f : \mathbb{R} \to \mathbb{R}$, differentiability implies continuity. As we will see here, the optimistic analogue for functions $f : \mathbb{R}^m \to \mathbb{R}$ does not hold, no matter how nice our function is along a line. Consider the function

$$\begin{aligned} f &: \mathbb{R}^2 \to \mathbb{R} \\ (x, y) &\mapsto \begin{cases} \frac{2x^3 y}{x^6+y^2} \text{ if } (x, y) \neq (0, 0), \\ 0 \text{ if } (x, y) = (0, 0). \end{cases} \end{aligned}$$

1. (20 pts) Show the restriction of f to any line through the origin is continuous at $(0, 0)$ i.e.

$$\lim_{t \to 0} f(ta, tb) = 0$$

for any $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$. You do not need to give an $\varepsilon - \delta$ proof, but be sure to justify why this limit exists.

Hint: Consider the case $b = 0$ separately.

2. (25 pts)Show that the directional derivative of f along a unit vector $\vec{v} = (u, v)$ is 0 at 0: $\partial_{\vec{v}} f(0,0) = 0$.

3. (15 pts) Show that despite f being continuous along every line, with a defined derivative in every direction, f is not continuous as a function of two variables at $(0,0)$

   Hint: Consider $\lim_{t\to 0} f(t, g(t))$ for some $g(t)$ a polynomial in t that vanishes at $t = 0$. What $g(t)$ causes the denominator to have a higher order of vanishing than the numerator?

Solution 1

1. We have that

$$f(ta, tb) = \frac{2a^3bt^4}{a^6t^6 + b^2t^2} = \frac{2a^3b}{a^6t^4 + b^2} \times t^2$$

   As $a^6t^4 + b^2 \geq b^2$ for all t, the first factor is bounded if $b \neq 0$, and hence

$$0 \leq |f(ta, tb)| \leq Ct^2$$

   for some constant C. As $\lim_{t\to 0} Ct^2 = 0$, we must have

$$\lim_{t\to 0} f(ta, tb) = 0 = f(0,0)$$

   and so the restriction of f to a line through the origin is continuous at $(0,0)$, unless that line is the x-axis. If $b = 0$, then $f(ta, tb) = 0$ for all $t \neq 0$, and so $\lim_{t\to 0} f(ta, tb) = 0$, and we still get continuity.

2. By definition

$$\partial_{\vec{v}} f(0,0) = \lim_{t\to 0} \frac{f(tu, tv) - f(0,0)}{t} = \lim_{t\to 0} \frac{2u^3v}{u^6t^4 + v^2} \times t$$

   If $v = 0$, then the difference quotient is equal to 0, and we obtain

$$\partial_{(1,0)} f(0,0) = 0.$$

Otherwise, as before, the first factor is bounded, as so by the squeeze theorem

$$\lim_{t\to 0} \left|\frac{f(tu,tv)-f(0,0)}{t}\right| = 0.$$

Hence $\partial_{\vec{v}} f(0,0) = 0$.

3. If we consider the limit along the path $(x,y) = (t,t^3)$, we see that

$$\lim_{t\to 0} f(t,t^3) = \lim_{t\to 0} \frac{2t^6}{t^6+t^6} = \lim_{t\to 0} 1 = 1 \neq f(0,0)$$

so we cannot have that f is continuous at $(0,0)$. To motivate why we take $y = t^3$, note that if $x = t$ and $y = t^k h(t)$ for some $h(t)$ with $h(0) \neq 0$, then the order of vanishing of the numerator is $3+k$, while the order of vanishing of the denominator is $\min(6, 2k)$. By choosing $k = 3$, these orders of vanishing become equal, and so we are guarenteed to obtain a finite, non-zero limit.

Exercise 2 *Deriving a special case of Lagrange multipliers (40pts)*

For purposes of this question, you should pretend that you have no pre-existing knowledge of Lagrange multipliers or optimisation with constraints. You may, however, freely use chain rule for functions of several variables.

1. (30 pts) Let

$$H^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 - y^2 = 1\}$$

be the standard hyperbola, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function with first order partial derivatives. Suppose that f, restricted to a function on the hyperbola, has a local extremum at a point $(x_0, y_0) \neq (1,0)$. By writing

$$H^1 = \{(\cosh(t), \sinh(t)) \mid t \in \mathbb{R}\}$$

show that there exists $\lambda \in \mathbb{R}$ such that

$$\frac{\partial f}{\partial x}(x_0,y_0) = 2x_0\lambda, \quad \frac{\partial f}{\partial y}(x_0,y_0) = -2y_0\lambda.$$

Hint: Since $x_0, y_0 \neq 0$, we can define λ satisfying one of these equations. Consider the function $F(t) = f(\cosh(t), \sinh(t))$. Freely using chain rule for such compositions, what does the derivative of F look like at a local extremum? Can we show that this implies λ satisfies the other?

2. (10 pts) Using this determine the possible local extrema of $f(x, y) = x^3 + 2y^3$ restricted to $x^2 - y^2 = 1$, and the associated λ.

   Remark: This means without using the parameterisation of H^1 in terms of hyperbolic trigonometric functions!

Solution 2

i) The function f restricted to H^1 can be viewed as a function on $\mathbb{R}$ by composing f with $(\cosh(t), \sinh(t))$. Define

$$F(t) := f(\cos(t), \sin(t)).$$

Choose $t_0 \in \mathbb{R}$ such that $(\cos(t_0), \sin(t_0)) = (x_0, y_0)$, we have that F has a local extremum at t_0. From the extreme value theorem, this implies that $F'(t_0) = 0$, as F is differentiable on $\mathbb{R}$. By the chain rule, this implies that

$$F'(t_0) = \frac{\partial f}{\partial x}(x_0, y_0)\sinh(t_0) + \frac{\partial f}{\partial y}(x_0, y_0)\cosh(t_0) = 0$$

which is to say that

$$\frac{\partial f}{\partial y}(x_0, y_0)x_0 = -\frac{\partial f}{\partial x}(x_0, y_0)y_0.$$

Since $y_0 \neq 0$, we can define $\lambda = \frac{-1}{2y_0}\frac{\partial f}{\partial y}(x_0, y_0)$, so that λ satisfies the second equation. We can then check that

$$\begin{aligned} 2x_0\lambda &= \frac{-x_0}{y_0}\frac{\partial f}{\partial y}(x_0, y_0) \\ &= \frac{1}{y_0}\left(y_0\frac{\partial f}{\partial x}(x_0, y_0)\right) = \frac{\partial f}{\partial x}(x_0, y_0). \end{aligned}$$

ii) From the previous question, any local extremum occurs at $(1, 0)$ or a point (x_0, y_0) such that

$$x_0^2 - y_0^2 = 1$$

and there exists λ

$$\frac{\partial f}{\partial x}(x_0, y_0) = 2x_0\lambda, \quad \frac{\partial f}{\partial y}(x_0, y_0) = -2y_0\lambda.$$

Thus, we want to solve the system

$$3x^2 = 2x\lambda, \quad 3y^2 = -4y\lambda, \quad x^2 - y^2 = 1.$$

For $x, y \neq 0$, we can solve this for x and y in terms of λ, which we can plug into the circle equation to determine λ. As $x = 0$ has no corresponding y, we can omit it. This gives us possible extrema at

$$\begin{aligned}(x, y, \lambda) &= (1, 0, \frac{3}{2}),\\ (x, y, \lambda) &= \pm\left(\frac{2}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \sqrt{3}\right)\end{aligned}$$

which give $x^3 + 2y^3 \in \{1, \pm\sqrt{3}^{-1})$.