# MAU22203/33203 - Analysis in Several Real Variables

Exercise Sheet 1

Trinity College Dublin

Course homepage

Answers are due for October $12^{\text{th}}$, 23:59.
The use of electronic calculators and computer algebra software is allowed.

## Exercise 1 *Properties of sequences (60pts)*

Let $\{\vec{x}_n\}$ and $\{\vec{y}_n\}$ be two sequences of points in $\mathbb{R}^m$, and let $\lambda \in \mathbb{R}$ be a real number. Suppose that $\{\vec{x}_n\}$ converges to a point $\vec{p}$, and $\{\vec{y}_n\}$ converges to a point $\vec{q}$. By giving a formal $\varepsilon$-$N$ proof, establish the following:

(15pts) The sequence $\{\lambda\vec{x}_n\}$ converges to $\lambda\vec{p}$,

*Hint: consider* $\lambda = 0$ *as a separate case.*

(15pts) The sequence $\{\vec{z}_n = \vec{x}_n + \vec{y}_n\}$ converges to $\vec{p} + \vec{q}$.

(30pts) Suppose further that, for all $n > 0$, $\|\vec{x}_n - \vec{y}_n\| < \frac{1}{n}$. Conclude that $\vec{p} = \vec{q}$.

*Hint: We haven't yet shown that limits commute with norms, so we can't freely conclude this. Can we show that* $\|\vec{p} - \vec{q}\|$ *must be smaller than all positive reals? Try the extending the triangle inequality to a sum of three terms!*

## Solution 1

1. First note that the claim is trivial if $\lambda = 0$, so we will assume otherwise. As $\{\vec{x}_n\}$ converges to $\vec{p}$, for every $\varepsilon > 0$, there exists $N > 0$ such that $\|\vec{x}_n - \vec{p}\| < \varepsilon$ for all $n \geq N$. Thus, for all $\varepsilon > 0$, there exists $N_\lambda > 0$ such that
$$\|\vec{x}_n - \vec{p}\| < \frac{\varepsilon}{|\lambda|}$$
for all $n \geq N_\lambda$. Hence, for all $n \geq N_\lambda$
$$\|\lambda\vec{x}_n - \lambda\vec{p}\| = |\lambda|\|\vec{x}_n - \vec{p}\| < |\lambda|\frac{\varepsilon}{|\lambda|} = \varepsilon$$
and so $\{\lambda\vec{x}_n\}$ converges to $\lambda\vec{p}$.

2. By the convergence of $\{\vec{x}_n\}$ and $\{\vec{y}_n\}$, we have that for every $\varepsilon > 0$ there exist $N_x, N_y > 0$ such that
$$\|\vec{x}_n - \vec{p}\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|\vec{y}_n - \vec{q}\| < \frac{\varepsilon}{2}$$
and so, by the triangle inequality
$$\begin{aligned}\|\vec{x}_n + \vec{y}_n - \vec{p} - \vec{q} &= \|(\vec{x}_n - \vec{p}) + (\vec{y}_n - \vec{q})\| \\ &\leq \|\vec{x}_n - \vec{p}\| + \|\vec{y}_n - \vec{q}\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$
for all $n \geq \max\{N_x, N_y\}$.

3. Note that if we can show $\|\vec{p} - \vec{q}\| < \varepsilon$ for all $\varepsilon > 0$, we must have that $\|\vec{p} - \vec{q}\| = 0$. This would imply that $\vec{p} - \vec{q} = 0$ and so $\vec{p} = \vec{q}$.

   So suppose we are given $\varepsilon > 0$. Then, note that, by applying the triangle inequality twice, we have that
$$\begin{aligned}\|\vec{p} - \vec{q}\| &= \|\vec{p} - \vec{x}_n + \vec{x}_n - \vec{y}_n + \vec{y}_n - \vec{q}\| \\ &\leq \|\vec{p} - \vec{x}_n\| + \|\vec{x}_n - \vec{y}_n\| + \vec{y}_n - \vec{q}\| \\ &= \|\vec{x}_n - \vec{p}\| + \|\vec{x}_n - \vec{y}_n\| + \vec{y}_n - \vec{q}\|.\end{aligned}$$
   As $\vec{x}_n \to \vec{p}$, there exists $N_1 > 0$ such that
$$\|\vec{x}_n - \vec{p}\| < \frac{\varepsilon}{3}$$

for all $n \geq N_1$. Similarly, there exists $N_2 > 0$ such that

$$\|\vec{y}_n - \vec{q}\| < \frac{\varepsilon}{3}$$

for all $n \geq N_2$. Finally, as $\frac{1}{n} \to 0$, there exists $N_3 > 0$ such that

$$\|\vec{x}_n - \vec{y}_n\| < \frac{1}{n} < \frac{\varepsilon}{3}$$

for all $n \geq N_3$. Thus, for all $n \geq N = \max(N_1, N_2, N_3)$, we have that

$$\|\vec{x}_n - \vec{p}\| + \|\vec{x}_n - \vec{y}_n\| + \vec{y}_n - \vec{q}\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

and so

$$\|\vec{p} - \vec{q}\| < \varepsilon$$

for all $\varepsilon > 0$. The claim then follows as explained above.

## **Exercise 2** *Bounded operators (40pts)*

In the following, you may use any standard facts from your first year courses. Let $A : \mathbb{R}^m \to \mathbb{R}^m$ be a linear transformation represented by the $(m \times m)$-matrix $(A_{i,j})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq m}}$ with respect to the standard bases.

(15 pts) Show that there exists a constant $C > 0$ such that

$$\|A\vec{x}\| \leq C\|\vec{x}|$$

for all $\vec{x}\| \in \mathbb{R}^m$.

*Hint: what are the components of $A\vec{x}$ and how could we bound them using the Cauchy-Schwarz inequality?*

(10 pts) Hence, or otherwise, show that

$$\|A^n\vec{x}\| \leq C^n\|\vec{x}\|$$

for all $\vec{x} \in \mathbb{R}^m$ and all $n \geq 0$.

(15 pts) Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be given by the below matrix, and define a sequence of points in $\mathbb{R}^3$ by $\vec{x}_n := A^{n-1}\vec{x}_1$, where $\vec{x}_1$ is given below. Prove that $\{\vec{x}_n\}$ converges to $\vec{0}$.

$$A = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{3} \\ 0 & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{5} & 0 & \frac{1}{20} \end{pmatrix}, \quad \vec{x}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

*Hint: Try to bound $\|\vec{x}_n - \vec{0}\| = \|\vec{x}_n\|$ by something that converges to 0.*

## Solution 2

1. Letting $\vec{v} = A\vec{x}$, note that

$$v_i = A_{i,1}x_1 + A_{i,2}x_2 + \cdots A_{i,m}x_m$$

is given by the inner product of the $i^{\text{th}}$ row of $A$ with $\vec{x}$. Hence, by the Cauchy-Schwarz inequality, we have that

$$v_i^2 = (A_{i,1}x_1 + A_{i,2}x_2 + \cdots A_{i,m}x_m)^2 \leq \left(A_{i,1}^2 + \cdots + A_{i,m}^2\right) \|\vec{x}\|^2$$

. Summing over $i$, we get that

$$\|\vec{v}\|^2 = \sum_{i=1}^{m} v_i^2 \leq \sum_{i=1}^{m} \left( \sum_{j=1}^{m} A_{i,j}^2 \right) \|\vec{x}\|^2$$

and so

$$\|A\vec{x}\|^2 = \|\vec{v}\|^2 \leq \left( \sum_{i=1}^{m} \sum_{j=1}^{m} A_{i,j}^2 \right) \|\vec{x}\|^2 = C^2 \|\vec{x}\|^2$$

from which the claim follows, taking

$$C = \sqrt{\left( \sum_{i=1}^{m} \sum_{j=1}^{m} A_{i,j}^2 \right)}.$$

2. This is true for $n = 1$, so assume it holds true for some $n > 1$. Then note that

$$\|A^{n+1}\vec{x}\| = \|A(A^n\vec{x})\| \leq C\|A^n\vec{x}\| \leq C^{n+1}\vec{x}$$

and so the claim follows by induction.

3. Note that we must have

$$\|\vec{x}_n\| = \|A^{n-1}\vec{x}_1\| \leq C^{n-1}\|\vec{x}_1\|$$

by the previous parts of the question. Taking $C$ as in the first part, we find

$$C \approx 0.8592\ldots < 1$$

and so

$$\|\vec{x}_n\| < (0.9)^{n-1}\|\vec{x}_1\|.$$

From first year analysis, we know that $(0.9)^n \to 0$, and hence the limit of

$$\|\vec{x}_n\| = \|\vec{x}_n - \vec{0}\|$$

is 0. Hence, we must have that $\{\vec{x}_n\}$ converges to $\vec{0}$.